IMA Journal of Numerical Analysis (2004) **24**, 671–698

A priori error estimates for a coupled finite element method and mixed finite element method for a fluid–solid interaction problem

XIAOBING FENG†
Department of Mathematics, The University of Tennessee, Knoxville TN 37996, USA

AND

ZHENGHUI XIE‡
Institute of Atmospheric Physics, Chinese Academy of Sciences, Beijing 100029, China

[Received on 26 March 2002; revised on 6 January 2003]

This paper presents a heterogeneous finite element method for a fluid–solid interaction problem. The method, which combines a standard finite element discretization in the fluid region and a mixed finite element discretization in the solid region, allows the use of different meshes in fluid and solid regions. Both semi-discrete and fully discrete approximations are formulated and analysed. Optimal order a priori error estimates in the energy norm are shown. The main difficulty in the analysis is caused by the two interface conditions which describe the interaction between the fluid and the solid. This is overcome by explicitly building one of the interface conditions into the finite element spaces. Iterative substructuring algorithms are also proposed for effectively solving the discrete finite element equations.

Keywords: acoustic and elastic waves; fluid–solid interaction; absorbing boundary condition; finite element and mixed finite element methods.

1. Introduction

Fluid–solid interaction problems have long been subjects of both theoretical and practical studies, and important applications of such problems are found in inverse scattering, elastoacoustics, geosciences, oceanography and the automobile industry. For some recent developments on modelling, mathematical analyses and numerical simulations, we refer to Demkowicz *et al.* (1991), Feng (2000), Feng *et al.* (2001), Santos *et al.* (1988) and the references therein.

The purpose of this paper is to develop a heterogeneous finite element method for a fluid–solid interaction model which was recently proposed in Feng *et al.* (2001). The heterogeneous method (in space) consists of standard Galerkin finite element discretizations in the fluid region and mixed finite element discretizations in the solid region which simultaneously approximate the stress and displacement variables. Fully discrete time-stepping schemes are also considered in the paper. Our main objective is to establish some optimal order error estimates in the energy norm of the fluid–solid

†Email: xfeng@math.utk.edu
‡Email: zxie@lasg.iap.ac.cn

interaction problem for both the semi-discrete and fully discrete heterogeneous finite element methods. The main difficulty for establishing the optimal order error estimates is caused by the interface conditions which describe the interaction between fluid and solid on their contact surface. We handle the difficulty by explicitly building one of the two interface conditions into the finite element spaces.

Numerical analysis for the fluid–solid interaction model and its variants have been studied by a number of authors. Demkowicz *et al.* (1991) proposed and analysed hp boundary element methods for the model without introducing absorbing boundary conditions. Feng (2000) developed and analysed the (standard) finite element method and some domain decomposition algorithms for the model. Makridakis *et al.* (1996) analysed the linear finite element method in one dimension for the model in the frequency domain. Santos *et al.* (1988) proposed and analysed a finite element method for the Boit model for propagation of low-frequency elastic waves in a fluid-saturated porous solid. See also Cowser *et al.* (1996), Dupont (1973), Makridakis (1992) and the references therein for detailed expositions on finite element and mixed finite element methods for acoustic and elastic wave equations.

The organization of this paper is as follows. In Section 2 we introduce some space notation, and state the fluid–solid interaction model and some basic facts about the model. In Section 3, we formulate a semi-discrete heterogeneous finite element approximation for the fluid–solid interaction model and establish an optimal order a priori error estimates in the energy norm. In Section 4 we propose a fully discrete heterogeneous finite element method by discretizing the semi-discrete method in time. An optimal order a priori error estimate is also established for the fully discrete method. In Section 5 we propose some parallelizable domain decomposition algorithms for solving the fully discrete finite element system.

2. The fluid–solid interaction model

We consider the propagation of waves in a composite medium Ω which consists of a fluid part Ω_f and a solid part Ω_s, that is $\Omega = \Omega_f \cup \Omega_s$. Ω will be identified with a domain in $\mathbf{R^N}$ for $N = 2, 3$, and will be taken to be of unit thickness when $N = 2$. Let $\Gamma = \partial\Omega_f \cap \partial\Omega_s$ denote the interface between two media, and let $\Gamma_f = \partial\Omega_f \setminus \Gamma$ and $\Gamma_s = \partial\Omega_s \setminus \Gamma$. The fluid–solid interaction model we are going to study in this paper is given by Feng *et al.* (2001) and Feng (2000):

$$\frac{1}{c^2} p_{tt} - \Delta p = g_f \qquad \text{in } \Omega_f \times (0, T), \tag{2.1}$$

$$\rho_s \underset{\sim}{u}_{tt} - \operatorname{div}\Big(\underset{\approx}{\sigma}\Big(\underset{\sim}{u}\Big)\Big) = \underset{\sim}{g}_s \qquad \text{in } \Omega_s \times (0, T), \tag{2.2}$$

$$\frac{\partial p}{\partial \mathbf{n}_f} - \rho_f \underset{\sim}{u}_{tt} \cdot \mathbf{n}_s = 0 \qquad \text{on } \Gamma \times (0, T), \tag{2.3}$$

$$\underset{\approx}{\sigma}\Big(\underset{\sim}{u}\Big)\mathbf{n}_s - p\mathbf{n}_f = 0 \qquad \text{on } \Gamma \times (0, T), \tag{2.4}$$

$$\frac{1}{c} p_t + \frac{\partial p}{\partial \mathbf{n}_f} = 0 \qquad \text{on } \Gamma_f \times (0, T), \tag{2.5}$$

$$\rho_s \underset{\approx}{\mathcal{A}_s} \underset{\sim}{u}_t + \underset{\approx}{\sigma}\Big(\underset{\sim}{u}\Big)\mathbf{n}_s = 0 \qquad \text{on } \Gamma_s \times (0, T), \tag{2.6}$$

$$p(x,0) = p_0(x), \quad p_t(x,0) = p_1(x) \qquad \text{in } \Omega_f, \tag{2.7}$$
$$\underset{\sim}{u}(x,0) = \underset{\sim}{u_0}(x), \quad \underset{\sim}{u_t}(x,0) = \underset{\sim}{u_1}(x) \qquad \text{in } \Omega_s, \tag{2.8}$$

where

$$\underset{\approx}{\sigma}\left(\underset{\sim}{u}\right) = \lambda_s \operatorname{div} \underset{\sim}{u}\, \underset{\approx}{I} + 2\mu_s \underset{\approx}{\varepsilon}\left(\underset{\sim}{u}\right), \qquad \underset{\approx}{\varepsilon}\left(\underset{\sim}{u}\right) = \frac{1}{2}\Big[\nabla \underset{\sim}{u} + (\nabla \underset{\sim}{u})^T\Big]. \tag{2.9}$$

In the above description, p is the pressure function in Ω_f and $\underset{\sim}{u}$ is the displacement vector in Ω_s. ρ_i $(i = f, s)$ denotes the density of Ω_i, $\mathbf{n}_i$ $(i = f, s)$ denotes the unit outward normal to $\partial\Omega_i$. $\lambda_s > 0$ and $\mu_s \geqslant 0$ are the Lamé constants of Ω_s. Equation (2.9) is the constitutive relation for Ω_s. $\underset{\approx}{I}$ stands for the $N \times N$ identity matrix. The boundary conditions in (2.5) and (2.6) are the first-order absorbing boundary conditions for acoustic and elastic waves, respectively. These boundary conditions are transparent to waves arriving normally at the boundary (cf. Engquist & Majda, 1979; Lysmer & Kuhlmeyer, 1969). $\underset{\approx}{\mathcal{A}_s}$ is an $N \times N$ symmetric positive definite constant matrix. Equations (2.3) and (2.4) are the interface conditions which describe the interaction between the fluid and the solid. For a detailed derivation of the above model and its analytical analysis, we refer to Feng *et al.* (2001) and Feng (2000).

The standard space notation is adopted in this paper. For example, $H^k(D)$, $k \geqslant 0$ integer, denotes the Sobolev spaces over the domain D. When $k = 0$, $H^0(D) = L^2(D)$, and $(\cdot,\cdot)_D$ is used to denote the standard inner product on $L^2(D)$. $\|\cdot\|_{k,D}$ denotes the usual norms on $H^k(D)$. For a Banach space B, $L^q(0,T;B)$ stands for the space of L^q–integrable functions with range in B. $W^{k,q}([0,T];B)$ is the space of functions whose up to kth-order derivatives with respect to t are in $L^q(0,T;B)$. $\underset{\sim}{B} = (B)^N$ and $\underset{\approx}{B} = (B)^{N\times N}$ stand for the vector and tensor spaces whose components are in space B, $\underset{\sim}{v}$ and $\underset{\approx}{\tau}$ denote arbitrary vectors and tensors in $\underset{\sim}{B} = (B)^N$ and $\underset{\approx}{B} = (B)^{N\times N}$, respectively. In addition, we introduce the following special space notation:

$$\mathcal{P}_f = \bigcap_{j=0}^{1} W^{j,\infty}(0,T;H^{1-j}(\Omega_f)) \cap \{p_t \in L^2(0,T;L^2(\Gamma_f))\},$$
$$\mathcal{Q}_f = \mathcal{P}_f \cap \bigcap_{j=1}^{2} W^{j,\infty}(0,T;H^{2-j}(\Omega_s)) \cap \{p_{tt} \in L^2(0,T;L^2(\Gamma_f))\},$$
$$\underset{\sim}{\mathcal{U}_s} = \bigcap_{j=0}^{1} W^{j,\infty}\Big(0,T;\underset{\sim}{H}^{1-j}(\Omega_s)\Big) \cap \Big\{\underset{\sim}{u_t} \in L^2\Big(0,T;\underset{\sim}{L}^2(\Gamma_s)\Big)\Big\},$$
$$\underset{\sim}{\mathcal{V}_s} = \mathcal{U}_s \cap \bigcap_{j=1}^{2} W^{j,\infty}\Big(0,T;\underset{\sim}{H}^{2-j}(\Omega_s)\Big) \cap \Big\{\underset{\sim}{u_{tt}} \in L^2\Big(0,T;\underset{\sim}{L}^2(\Gamma_s)\Big)\Big\},$$
$$\widehat{\mathcal{Q}}_f = Q_f \cap L^\infty(0,T;H^2(\Omega_f)), \qquad \underset{\sim}{\widehat{\mathcal{V}}_s} = \underset{\sim}{\mathcal{V}_s} \cap L^\infty\Big(0,T;\underset{\sim}{H}^2(\Omega_s)\Big),$$
$$\underset{\approx}{H_s} = \Big\{\underset{\approx}{\sigma} \in \underset{\approx}{L}^2(\Omega_s);\ \operatorname{div}\underset{\approx}{\sigma} \in \underset{\sim}{L}^2(\Omega_s),\ \underset{\approx}{\sigma} = \underset{\approx}{\sigma}^T\Big\},$$
$$\mathbf{W} = \Big\{\Big(p,\underset{\sim}{u},\underset{\approx}{\sigma}\Big) \in H^1(\Omega_f) \times \underset{\sim}{L}^2(\Omega_s) \times \underset{\approx}{H_s};\ \underset{\approx}{\sigma}\mathbf{n}_s - p\mathbf{n}_f = 0 \text{ on } \Gamma\Big\},$$

where $(\operatorname{div}\underset{\approx}{\sigma})_i = \sum_{j=1}^{N} \partial_j \sigma_{ij}$ for $i = 1, \ldots, N$.

We shall make the following physical and mathematical assumptions throughout the paper. The same assumptions were made in Feng *et al.* (2001) and Feng (2000).

Assumption A:

- (A1) $\rho_f = \text{constant} > 0, \quad \infty > \underline{\rho_s} \geqslant \rho_s = \rho_s(x) \geqslant \overline{\rho_s} > 0\cdot$ λ_s, μ_s are all positive constants.
- (A2) $\Omega_f \subset \mathbf{R}^N$, $\Omega_s \subset \mathbf{R}^N$ for $N = 2, 3$ are bounded open sets with Lipschitz continuous boundary $\partial\Omega_f$ and $\partial\Omega_s$, respectively.
- (A3) $\overline{\Omega} = \overline{\Omega}_f \cup \overline{\Omega}_s$, meas$(\Omega_f) \neq 0$, meas$(\Omega_s) \neq 0$. Assume that $\Gamma \neq \emptyset$. Note that it is acceptable if one of Γ_f and Γ_s is empty.
- (A4) Suppose that the initial datum functions satisfy the following compatibility conditions.

Compatibility Condition C:

- (C1) $\underset{\sim}{u_0} \in \underset{\sim}{H}^2(\Omega_s)$ and $\underset{\sim}{u_1} \in \underset{\sim}{H}^1(\Omega_s)$ are said to be compatible on Γ_s if $\rho_s \underset{\approx}{\mathcal{A}_s} \underset{\sim}{u_1} + \underset{\approx}{\sigma}(\underset{\sim}{u_0})\mathbf{n}_s = 0$, on Γ_s.
- (C2) $p_0 \in H^1(\Omega_f)$ and $\underset{\sim}{u_0} \in \underset{\sim}{H}^2(\Omega_s)$ are said to be compatible on Γ if $\underset{\approx}{\sigma}(\underset{\sim}{u_0})\mathbf{n}_s - p_0\mathbf{n}_f = 0$, on Γ.

Under the above assumptions, the following existence, uniqueness and regularity results were established in Feng *et al.* (2001) for the fluid–solid interaction model (2.1)–(2.9).

THEOREM 1 The initial-boundary value problem (2.1)–(2.9) has a unique weak solution $(p, \underset{\sim}{u}) \in \mathcal{P}_f \times \mathcal{V}_s$ (in the distribution sense). Moreover, if Ω_f and Ω_s are convex polygon or polyhedron domains, then $(p, \underset{\sim}{u}) \in \widehat{\mathcal{Q}}_f \times \widehat{\mathcal{V}}_s$; and both p and $\underset{\sim}{u}$ are smooth in t variable if the source functions are smooth in t variable,

Next we will derive an equivalent mixed formulation for (2.1)–(2.9). This will be done by introducing the stress tensor $\underset{\approx}{\sigma} = \underset{\approx}{\sigma}(\underset{\sim}{u})$ as an additional independent unknown variable so that p, $\underset{\sim}{u}$ and $\underset{\approx}{\sigma}$ will be determined simultaneously by solving the mixed formulation of the problem. For more exposition on the theory of mixed finite element methods, we refer to Brezzi & Fortin (1991).

Applying the matrix trace operator tr to both sides of (2.9) and solving for $\mathrm{tr}(\underset{\approx}{\varepsilon}(\underset{\sim}{u}))$, we have

$$\frac{\lambda_s}{2\mu_s}\,\mathrm{tr}\Big(\underset{\approx}{\varepsilon}\Big(\underset{\sim}{u}\Big)\Big) = \gamma_s\,\mathrm{tr}\Big(\underset{\approx}{\sigma}\Big), \quad \text{where } \gamma_s = \begin{cases} \dfrac{\lambda_s}{4\mu_s(\lambda_s + \mu_s)}, & \text{if } N = 2,\\[2mm] \dfrac{\lambda_s}{2\mu_s(3\lambda_s + 2\mu_s)}, & \text{if } N = 3. \end{cases} \tag{2.10}$$

Substituting (2.10) into (2.9) and then differentiating the equation with respect to t we get

$$\frac{1}{2\mu_s}\underset{\approx}{\sigma_t} - \gamma_s\,\mathrm{tr}\Big(\underset{\approx}{\sigma_t}\Big)I - \underset{\approx}{\varepsilon}\Big(\underset{\sim}{u_t}\Big) = 0. \tag{2.11}$$

Now testing (2.1), (2.2) and (2.11), and integrating by parts and using the boundary and interface conditions we conclude that the unique weak solution to problem (2.1)–(2.9) (cf. Theorem 1) $(p, \underset{\sim}{u}, \underset{\approx}{\sigma}) : [0, T] \to \mathbf{W}$ satisfies for any $(q, \underset{\sim}{v}, \underset{\approx}{\chi}) \in \mathbf{W}$

$$\left(\frac{1}{c^2} p_{tt}, q\right)_{\Omega_f} + (\nabla p, \nabla q)_{\Omega_f} + \left\langle \frac{1}{c} p_t, q \right\rangle_{\Gamma_f} - \left\langle \rho_f \underset{\sim}{u_{tt}} \cdot \mathbf{n}_s, q \right\rangle_{\Gamma} = (g_f, q)_{\Omega_f}, \tag{2.12}$$

$$a\left(\underset{\approx}{\sigma_t}, \underset{\approx}{\chi}\right) + \left(\underset{\sim}{u_t}, \operatorname{div} \underset{\approx}{\chi}\right)_{\Omega_s} + \left\langle \left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1} \underset{\approx}{\sigma} \mathbf{n}_s, \underset{\approx}{\chi} \mathbf{n}_s \right\rangle_{\Gamma_s} - \left\langle \underset{\sim}{u_t}, \underset{\approx}{\chi} \mathbf{n}_s \right\rangle_{\Gamma} = 0, \tag{2.13}$$

$$\left(\rho_s \underset{\sim}{u_{tt}}, \underset{\sim}{v}\right)_{\Omega_s} - \left(\operatorname{div} \underset{\approx}{\sigma}, \underset{\sim}{v}\right)_{\Omega_s} = \left(\underset{\sim}{g_s}, \underset{\sim}{v}\right)_{\Omega_s}, \tag{2.14}$$

$$\underset{\approx}{\sigma_0}(x) := \underset{\approx}{\sigma}(x, 0) = \lambda_s \operatorname{tr}\left(\underset{\approx}{\varepsilon}\left(\underset{\sim}{u_0}(x)\right)\right) + 2\mu_s \underset{\approx}{\varepsilon}\left(\underset{\sim}{u_0}(x)\right) \qquad \forall x \in \Omega, \tag{2.15}$$

where

$$a\left(\underset{\approx}{\sigma}, \underset{\approx}{\chi}\right) = \left(\frac{1}{2\mu_s} \underset{\approx}{\sigma}, \underset{\approx}{\chi}\right)_{\Omega_s} - \gamma_s \left(\operatorname{tr}\left(\underset{\approx}{\sigma}\right), \operatorname{tr}\left(\underset{\approx}{\chi}\right)\right)_{\Omega_s}. \tag{2.16}$$

LEMMA 1 The problem defined by (2.12)–(2.16), (2.7) and (2.8) has a unique solution.

Proof. The existence follows from Theorem 1. To show the uniqueness, it suffices to show the system only has the trivial solution if $g_f = 0$, $\underset{\sim}{g_s} = 0$, $\underset{\sim}{u_0} = 0$, (hence, $\underset{\approx}{\sigma_0} = 0$), $\underset{\sim}{u_1} = 0$, $p_0 = 0$ and $p_1 = 0$.

After integrating (2.12) with respect to t, setting $q = p$, $\underset{\approx}{\chi} = \rho_f \underset{\approx}{\sigma}$ in (2.13) and $\underset{\sim}{v} = \rho_f \underset{\sim}{u_t}$ in (2.14), and adding the resulting equations we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \left[\left\| \frac{1}{c} p \right\|_{0,\Omega_f}^2 + \left\| \nabla \int_0^t p(s) \mathrm{d}s \right\|_{0,\Omega_f}^2 + \rho_f a\left(\underset{\approx}{\sigma}, \underset{\approx}{\sigma}\right) + \left\| \sqrt{\rho_f \rho_s} \underset{\sim}{u_t} \right\|_{0,\Omega_s}^2 \right] + \left\| \frac{1}{\sqrt{c}} p \right\|_{0,\Gamma_f}^2 + \rho_f \left\langle \left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1} \underset{\approx}{\sigma} \mathbf{n}_s, \underset{\approx}{\sigma} \mathbf{n}_s \right\rangle_{\Gamma_s} = 0, \tag{2.17}$$

where we have used the fact that $(p, \underset{\sim}{u}, \underset{\approx}{\sigma}) \in \mathbf{W}$.

Since each term in (2.17) is non-negative (see Lemma 3.2 of Arnold *et al.*, 1984 for the proof of the positivity of the bilinear form $a(\cdot, \cdot)$), the zero initial data immediately implies that $p = 0$ a.e. in $\Omega_f \times (0, T)$ and $\underset{\sim}{u} = 0$ a.e. in $\Omega_s \times (0, T)$. The proof is complete. □

We remark that (2.12)–(2.16) and (2.7)–(2.8) is called a mixed formulation for problem (2.1)–(2.9). In Sections 3 and 4, we will construct and analyse some semi-discrete and fully discrete finite element methods for problem (2.1)–(2.9) based on this formulation.

3. Semi-discrete finite element approximations

In this section we shall formulate semi-discrete mixed finite element approximations for the initial-boundary value problem (2.1)–(2.9), and derive *a priori* error estimates under certain assumptions on the approximate starting values and on the smoothness of the solution. Throughout the rest of this paper, unless stated otherwise, C and C_0 will denote a general positive constant, not necessarily the same in any two places.

3.1 *Formulation of semi-discrete finite element methods*

For simplicity of presentation, we assume that both Ω_f and Ω_s are polygonal domains. Let $\mathcal{T}_h$ be a quasi-uniform 'triangulation' of $\overline{\Omega}_f \cup \overline{\Omega}_s$ with the mesh size $h > 0$. Although it is not necessary, to avoid some complicated technicalities, we assume that $\mathcal{T}_h$ results in matched grids on the interface Γ: that is, no vertex of finite elements of one subdomain lies in the inside of an edge/face of a finite element from the other subdomain on the interface Γ. Let $P_f^h \subset H^1(\Omega_f)$ be a (Lagrange) finite element space consisting of continuous piecewise polynomials of degree k $(k \geqslant 1)$ on $\mathcal{T}_h|_{\Omega_f}$, and $\underset{\sim}{V}_s^h \times \underset{\approx}{H}_s^h \subset \underset{\sim}{L}^2(\Omega_s) \times \underset{\approx}{H}_s$ be a pair of stable mixed finite element subspaces on $\mathcal{T}_h|_{\Omega_s}$ for the linear elasticity problems. Several choices of $\underset{\sim}{V}_s^h \times \underset{\approx}{H}_s^h$ are known to be acceptable (cf. Brezzi & Fortin, 1991). In this paper we only consider the family of subspaces due to Arnold *et al.* (1984), which were constructed using composite elements. Specifically, we let $\underset{\sim}{V}_s^h \times \underset{\approx}{H}_s^h$ denote the Arnold–Douglas–Gupta element of order k $(k \geqslant 2)$, which means that the components of $\underset{\sim}{V}_s^h$ are polynomials of degree $k-1$ on each triangle of $\mathcal{T}_h|_{\Omega_s}$ and components of $\underset{\approx}{H}_s^h$ are piecewise kth-order polynomials on each triangle. Recall that Arnold–Douglas–Gupta elements satisfy the inclusion $\operatorname{div} \underset{\approx}{H}_s^h \subset \underset{\sim}{V}_s^h$.

Define the space

$$\mathbf{W}^h = \left\{ \left(q_h, \underset{\sim}{v}_h, \underset{\approx}{\tau}_h\right) \in P_f^h \times \underset{\sim}{V}_s^h \times \underset{\approx}{H}_s^h;\ \underset{\approx}{\tau}_h \mathbf{n}_s - q_h \mathbf{n}_f = 0 \text{ pointwise on } \Gamma \right\}.$$

We remark that the constraint on Γ in the definition of $\mathbf{W}^h$ can be fulfilled by requiring the nodal parameters of $q_h \mathbf{n}_f$ and the nodal parameters of $\underset{\approx}{\tau}_h \mathbf{n}_s$ on Γ to be the same since the values of $\underset{\approx}{\tau}_h \mathbf{n}_s$ at the nodes on triangle edges are used as the degrees of freedom for constructing the space $\underset{\sim}{V}_s^h \times \underset{\approx}{H}_s^h$ (cf. p. 15 of Arnold *et al.*, 1984 and Section 5).

From Ciarlet (1978) and Arnold *et al.* (1984) we know that for any $(q, \underset{\sim}{v}, \underset{\approx}{\tau}) \in H^1(\Omega_f) \times \underset{\sim}{L}^2(\Omega_s) \times \underset{\approx}{H}_s \cap H^{k+\frac{3}{2}}(\Omega_f) \times \underset{\sim}{H}^k(\Omega_s) \times \underset{\approx}{H}^{k+\frac{3}{2}}(\Omega_s)$, there exists $(q_h, \underset{\sim}{v}_h, \underset{\approx}{\tau}_h) \in P_f^h \times \underset{\sim}{V}_s^h \times \underset{\approx}{H}_s^h$ such that

$$\|q - q_h\|_{0,\Omega_f} + h\|\nabla(q - q_h)\|_{0,\Omega_f} \leqslant Ch^j \|q\|_{j,\Omega_f}, \quad 1 \leqslant j \leqslant k+1, \tag{3.1}$$

$$\|q - q_h\|_{0,\Gamma_f} \leqslant Ch^j \|q\|_{j,\Gamma_f}, \quad 1 \leqslant j \leqslant k+1, \tag{3.2}$$

$$\left\|\underset{\sim}{v} - \underset{\sim}{v}_h\right\|_{0,\Omega_s} \leqslant Ch^j \left\|\underset{\sim}{v}\right\|_{j,\Omega_s}, \quad 1 \leqslant j \leqslant k, \tag{3.3}$$

$$\left\|\underset{\approx}{\tau} - \underset{\approx}{\tau}_h\right\|_{0,\Omega_s} + h\left\|\operatorname{div}\left(\underset{\approx}{\tau} - \underset{\approx}{\tau}_h\right)\right\|_{0,\Omega_s} \leqslant Ch^j \left\|\underset{\approx}{\tau}\right\|_{j,\Omega_s}, \quad 1 \leqslant j \leqslant k+1, \tag{3.4}$$

$$\left\|\left(\underset{\approx}{\tau} - \underset{\approx}{\tau}_h\right)\mathbf{n}_s\right\|_{0,\Gamma_s} \leqslant Ch^j \left\|\underset{\approx}{\tau}\right\|_{j,\Gamma_s}, \quad 1 \leqslant j \leqslant k+1. \tag{3.5}$$

Now for any $(q, \underset{\sim}{v}, \underset{\approx}{\tau}) \in \mathbf{W} \cap H^{k+\frac{3}{2}}(\Omega_f) \times \underset{\sim}{H}^k(\Omega_s) \times \underset{\approx}{H}^{k+\frac{3}{2}}(\Omega_s)$, first, choose $\underset{\sim}{v}_h :=$

$Q_h \underset{\sim}{v}$, the L^2-projection of $\underset{\sim}{v}$ into $\underset{\sim}{V_s^h}$ defined by

$$\left(\underset{\sim}{v} - Q_h \underset{\sim}{v}, \underset{\sim}{w_h}\right)_{\Omega_s} = 0 \qquad \forall \underset{\sim}{w_h} \in \underset{\sim}{V_s^h}. \tag{3.6}$$

It is well known that there holds (Ciarlet, 1978)

$$\left\|\underset{\sim}{v} - Q_h \underset{\sim}{v}\right\|_{0,\Omega_s} \leqslant C h^j \left\|\underset{\sim}{v}\right\|_{j,\Omega_f}, \quad 1 \leqslant j \leqslant k. \tag{3.7}$$

Second, let $\underset{\approx}{\tau_h} := \pi_h \underset{\approx}{\tau}$, where π_h is the projection operator from $\underset{\approx}{H_s}$ to $\underset{\approx}{H_s^h}$ defined in Arnold *et al.* (1984) (see pp. 5, 15, 18 and 19 of Arnold *et al.*, 1984). Hence, (3.4) and (3.5) hold. Finally, let $q_h \in P_f^h$ be the elliptic projection of q defined by

$$(\nabla(q_h - q), \nabla\psi_h)_{\Omega_f} = 0 \qquad \forall \psi_h \in P_f^h \cap H_\Gamma^1(\Omega_f), \tag{3.8}$$

$$\langle q_h - q, \eta_h \rangle_\Gamma = 0 \qquad \forall \eta_h \in M_\Gamma^h, \tag{3.9}$$

where $H_\Gamma^1(\Omega_f) := \{\psi \in H^1(\Omega_f); \psi|_\Gamma = 0\}$, $\langle \cdot, \cdot \rangle_\Gamma$ denotes the L^2-inner product on Γ, and M_Γ^h is the space of all piecewise polynomials of degree k on Γ. From the finite element theory for elliptic problems (cf. Babuška, 1973 and Chapter 5 of Brenner & Scott, 1994) we know that such a q_h exists and satisfies the estimates (3.1) and (3.2), particularly when Ω_f is a smooth or convex polygonal domain.

Recall that π_h defined in Arnold *et al.* (1984) satisfies

$$\int_e \left(\underset{\approx}{\tau} - \underset{\approx}{\tau_h}\right)\mathbf{n}_s \cdot \underset{\sim}{\eta_h}\, \mathrm{d}s = \int_e \left(\underset{\approx}{\tau} - \pi_h \underset{\approx}{\tau}\right)\mathbf{n}_s \cdot \underset{\sim}{\eta_h}\, \mathrm{d}s = 0 \qquad \forall \underset{\sim}{\eta_h} \in [P_k(e)]^N \tag{3.10}$$

for each edge e of the triangulation $\mathcal{T}_h$. For $(q, \underset{\sim}{v}, \underset{\approx}{\tau}) \in \mathbf{W} \cap H^{k+\frac{3}{2}}(\Omega_f) \times \underset{\sim}{H}^k(\Omega_s) \times \underset{\approx}{H}^{k+\frac{3}{2}}(\Omega_s)$, let $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\tau_h})$ be chosen as above, then it follows from (3.9), (3.10) and the definition of $\mathbf{W}$ that

$$\begin{aligned}\int_\Gamma \left(q_h \mathbf{n}_f - \underset{\approx}{\tau_h}\mathbf{n}_s\right) \cdot \underset{\sim}{\eta_h}\, \mathrm{d}s &= \int_\Gamma (q_h - q)\mathbf{n}_f \cdot \underset{\sim}{\eta_h}\, \mathrm{d}s + \int_\Gamma \left(q\mathbf{n}_f - \underset{\approx}{\tau}\mathbf{n}_s\right) \cdot \underset{\sim}{\eta_h}\, \mathrm{d}s \\ &\quad + \int_\Gamma \left(\underset{\approx}{\tau} - \underset{\approx}{\tau_h}\right)\mathbf{n}_s \cdot \underset{\sim}{\eta_h}\, \mathrm{d}s = 0, \qquad \forall \underset{\sim}{\eta_h} \in [M_\Gamma^h]^N.\end{aligned}$$

Since $(q_h \mathbf{n}_f - \underset{\approx}{\tau_h}\mathbf{n}_s)|_\Gamma \in [M_\Gamma^h]^N$, hence, $\underset{\approx}{\tau_h}(x)\mathbf{n}_s \equiv q_h(x)\mathbf{n}_f$, $\forall x \in \Gamma$, consequently, $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\tau_h}) \in \mathbf{W}^h$. In summary, we have proved the following lemma.

LEMMA 2 For any $(q, \underset{\sim}{v}, \underset{\approx}{\tau}) \in \mathbf{W} \cap H^{k+\frac{3}{2}}(\Omega_f) \times \underset{\sim}{H}^k(\Omega_s) \times \underset{\approx}{H}^{k+\frac{3}{2}}(\Omega_s)$, there exists $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\tau_h}) \in \mathbf{W}^h$ such that (3.1)–(3.5) hold.

Based on the weak formulation (2.12)–(2.16), we define our semi-discrete mixed finite element method as: Find $(P, \underset{\sim}{U}, \underset{\approx}{\Pi}) : [0, T] \to \mathbf{W}^h$ such that for any $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h}) \in \mathbf{W}^h$

$$\left(\frac{1}{c^2} P_{tt}, q_h\right)_{\Omega_f} + (\nabla P, \nabla q_h)_{\Omega_f} + \left\langle \frac{1}{c} P_t, q_h \right\rangle_{\Gamma_f} - \left\langle \rho_f \underset{\sim}{U_{tt}} \cdot \mathbf{n}_s, q_h \right\rangle_{\Gamma} = (g_f, q_h)_{\Omega_f}, \tag{3.11}$$

$$a\left(\underset{\approx}{\Pi_t}, \underset{\approx}{\chi_h}\right) + \left(\underset{\sim}{U_t}, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s} + \left\langle \left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1} \underset{\approx}{\Pi} \mathbf{n}_s, \underset{\approx}{\chi_h} \mathbf{n}_s \right\rangle_{\Gamma_s} - \left\langle \underset{\sim}{U_t}, \underset{\approx}{\chi_h} \mathbf{n}_s \right\rangle_{\Gamma} = 0, \tag{3.12}$$

$$\left(\rho_s \underset{\sim}{U_{tt}}, \underset{\sim}{v_h}\right)_{\Omega_s} - \left(\operatorname{div} \underset{\approx}{\Pi}, \underset{\sim}{v_h}\right)_{\Omega_s} = (\underset{\sim}{g_s}, \underset{\sim}{v_h})_{\Omega_s}, \tag{3.13}$$

$$P(0) = P_0, \quad P_t(0) = P_1, \quad \underset{\sim}{U}(0) = \underset{\sim}{U_0}, \quad \underset{\sim}{U_t}(0) = \underset{\sim}{U_1}, \quad \underset{\approx}{\Pi}(0) = \underset{\approx}{\Pi_0}, \tag{3.14}$$

where P_0, P_1, $\underset{\sim}{U_0}$, $\underset{\sim}{U_1}$ and $\underset{\approx}{\Pi_0}$ are some approximate starting values which will be specified in the next section.

REMARK 1 Since the problem (3.11)–(3.14) can be rewritten as a linear system of second-order ordinary differential equations which has same number of unknowns and equations, its existence follows from standard ordinary differential equation theory (Hale, 1969). The uniqueness can be proved by exactly following the proof of Lemma 1.

3.2 *A priori error estimates*

Throughout the rest of this section, $(p, \underset{\sim}{u}, \underset{\approx}{\sigma})$ denotes the weak solution to (2.12)–(2.16), (2.7) and (2.8) as described in Lemma 1. $(P, \underset{\sim}{U}, \underset{\approx}{\Pi})$ denotes the solution to (3.11)–(3.14).

LEMMA 3 Let $r = p - P$, $\underset{\sim}{e} = \underset{\sim}{u} - \underset{\sim}{U}$ and $\underset{\approx}{E} = \underset{\approx}{\sigma} - \underset{\approx}{\Pi}$, there exists a constant $C > 0$ such that there holds the inequality

$$\begin{aligned}
&\|r_t\|^2_{L^\infty(L^2(\Omega_f))} + \|r\|^2_{L^\infty(H^1(\Omega_f))} + \|r_t\|^2_{L^2(L^2(\Gamma_f))} \\
&\qquad + \left\|\underset{\sim}{e_{tt}}\right\|^2_{L^\infty(L^2(\Omega_s))} + \left\|\underset{\approx}{E_t}\right\|^2_{L^\infty(L^2(\Omega_s))} + \left\|\underset{\approx}{E_t}\mathbf{n}_s\right\|^2_{L^2(L^2(\Gamma_s))} \\
&\leqslant C \int_0^\tau (\|\nabla(\hat{q}_t - p_t)\|^2_{0,\Omega_f} + \|\hat{q}_{tt} - p_{tt}\|^2_{0,\Omega_f} + \|\hat{q}_t - p_t\|^2_{0,\Gamma_f})\mathrm{d}t \\
&\quad + C \int_0^\tau \left(\left\|\underset{\sim}{\hat{v}_{tt}} - \underset{\sim}{u_{tt}}\right\|^2_{0,\Omega_s} + \left\|\left(Q_h \underset{\sim}{g_s} - \underset{\sim}{g_s}\right)_t\right\|^2_{0,\Omega_s}\right)\mathrm{d}t \\
&\quad + C \int_0^\tau \left(\left\|\underset{\approx}{\hat{\chi}_{tt}} - \underset{\approx}{\sigma_{tt}}\right\|^2_{0,\Omega_s} + \left\|\underset{\approx}{\hat{\chi}_t} - \underset{\approx}{\sigma_t}\right\|^2_{0,\Gamma_s} + \left\|\operatorname{div}\left(\underset{\approx}{\hat{\chi}_{tt}} - \underset{\approx}{\sigma_{tt}}\right)\right\|^2_{0,\Omega_s}\right)\mathrm{d}t \\
&\quad + C\left(\left\|r_t(0)\right\|^2_{L^2(\Omega_f)} + \left\|\underset{\approx}{E_t}(0)\right\|^2_{L^2(\Omega_s)} + \left\|r(0)\right\|^2_{H^1(\Omega_f)} + \left\|\underset{\sim}{e_{tt}}(0)\right\|^2_{L^2(\Omega_s)}\right)
\end{aligned} \tag{3.15}$$

for any $(\hat{q}, \underset{\sim}{\hat{v}}, \underset{\approx}{\hat{\chi}}) \in H^2(0, T; \mathbf{W}^h)$.

Proof. Differentiating (2.13) with respect to t yields

$$a\Big(\underset{\approx}{\sigma}_{tt}, \underset{\approx}{\chi}\Big) + \Big(\underset{\sim}{u}_{tt}, \operatorname{div} \underset{\approx}{\chi}\Big)_{\Omega_s} + \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{\sigma}_t \mathbf{n}_s, \underset{\approx}{\chi} \mathbf{n}_s\Big\rangle_{\Gamma_s} - \Big\langle \underset{\sim}{u}_{tt}, \underset{\approx}{\chi} \mathbf{n}_s\Big\rangle_{\Gamma} = 0. \tag{3.16}$$

Since $(q, \underset{\sim}{v}, \underset{\approx}{\chi}) \in \mathbf{W}$, we have from (2.12), (2.14) and (3.16) that

$$\begin{aligned} &\left(\frac{1}{c^2} p_{tt}, q\right)_{\Omega_f} + (\nabla p, \nabla q)_{\Omega_f} + \rho_f a\Big(\underset{\approx}{\sigma}_{tt}, \underset{\approx}{\chi}\Big) + \rho_f \Big(\underset{\sim}{u}_{tt}, \operatorname{div} \underset{\approx}{\chi}\Big)_{\Omega_s} \\ &\qquad + \left\langle \frac{1}{c} p_t, q\right\rangle_{\Gamma_f} + \rho_f \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{\sigma}_t \mathbf{n}_s, \underset{\approx}{\chi} \mathbf{n}_s\Big\rangle_{\Gamma_s} = (g_f, q)_{\Omega_f}, \end{aligned} \tag{3.17}$$

$$\Big(\rho_s \underset{\sim}{u}_{tt}, \underset{\sim}{v}\Big)_{\Omega_s} - \Big(\operatorname{div} \underset{\approx}{\sigma}, \underset{\sim}{v}\Big)_{\Omega_s} = \Big(\underset{\sim}{g}_s, \underset{\sim}{v}\Big)_{\Omega_s}. \tag{3.18}$$

Similarly, we have for any $(q_h, \underset{\sim}{v}_h, \underset{\approx}{\chi}_h) \in \mathbf{W}^h$

$$\begin{aligned} &\left(\frac{1}{c^2} P_{tt}, q_h\right)_{\Omega_f} + (\nabla P, \nabla q_h)_{\Omega_f} + \rho_f a\Big(\underset{\approx}{\Pi}_{tt}, \underset{\approx}{\chi}_h\Big) + \rho_f \Big(\underset{\sim}{U}_{tt}, \operatorname{div} \underset{\approx}{\chi}_h\Big)_{\Omega_s} \\ &\qquad + \left\langle \frac{1}{c} P_t, q_h\right\rangle_{\Gamma_f} + \rho_f \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{\Pi}_t \mathbf{n}_s, \underset{\approx}{\chi}_h \mathbf{n}_s\Big\rangle_{\Gamma_s} = (g_f, q_h)_{\Omega_f}, \end{aligned} \tag{3.19}$$

$$\Big(\rho_s \underset{\sim}{U}_{tt}, \underset{\sim}{v}_h\Big)_{\Omega_s} - \Big(\operatorname{div} \underset{\approx}{\Pi}, \underset{\sim}{v}_h\Big)_{\Omega_s} = \Big(\underset{\sim}{g}_s, \underset{\sim}{v}_h\Big)_{\Omega_s}. \tag{3.20}$$

Since $\mathbf{W}^h \subset W$, for any $(q_h, \underset{\sim}{v}_h, \underset{\approx}{\chi}_h) \in \mathbf{W}^h$, subtracting (3.19) from (3.17) and (3.20) from (3.18) we get the following error equations:

$$\begin{aligned} &\left(\frac{1}{c^2} r_{tt}, q_h\right)_{\Omega_f} + (\nabla r, \nabla q_h)_{\Omega_f} + \rho_f \Big(\underset{\sim}{e}_{tt}, \operatorname{div} \underset{\approx}{\chi}_h\Big)_{\Omega_s} + \rho_f a\Big(\underset{\approx}{E}_{tt}, \underset{\approx}{\chi}_h\Big) \\ &\qquad + \left\langle \frac{1}{c} r_t, q_h\right\rangle_{\Gamma_f} + \rho_f \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{E}_t \mathbf{n}_s, \underset{\approx}{\chi}_h \mathbf{n}_s\Big\rangle_{\Gamma_s} = 0, \end{aligned} \tag{3.21}$$

$$\Big(\rho_s \underset{\sim}{e}_{tt}, \underset{\sim}{v}_h\Big)_{\Omega_s} - \Big(\operatorname{div} \underset{\approx}{E}, \underset{\sim}{v}_h\Big)_{\Omega_s} = 0. \tag{3.22}$$

For any $(\hat{q}, \underset{\sim}{\hat{v}}, \underset{\approx}{\hat{\chi}}) : [0, T] \to \mathbf{W}^h$, set $(q_h, \underset{\sim}{v}_h, \underset{\approx}{\chi}_h)$ in (3.21) and (3.22) as follows:

$$q_h = r_t + (\hat{q}_t - p_t), \quad \underset{\sim}{v}_h = \underset{\sim}{e}_{tt} + \Big(\underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\Big), \quad \underset{\approx}{\chi}_h = \underset{\approx}{E}_t + \Big(\underset{\approx}{\hat{\chi}}_{tt} - \underset{\approx}{\sigma}_t\Big).$$

We then have

$$
\begin{aligned}
&\left(\frac{1}{c^2}r_{tt}, r_t\right)_{\Omega_f} + (\nabla r, \nabla r_t)_{\Omega_f} + \rho_f\left(\underset{\sim}{e}_{tt}, \operatorname{div}\underset{\approx}{E}_t\right)_{\Omega_s} \\
&\qquad + \rho_f a\left(\underset{\approx}{E}_{tt}, \underset{\approx}{E}_t\right) + \left\langle\frac{1}{c}r_t, r_t\right\rangle_{\Gamma_f} + \rho_f\left\langle\left(\rho_s\underset{\approx}{\mathcal{A}_s}\right)^{-1}\underset{\approx}{E}_t\mathbf{n}_s, \underset{\approx}{E}_t\mathbf{n}_s\right\rangle_{\Gamma_s} \\
&\qquad + \left(\frac{1}{c^2}r_{tt}, \hat{q}_t - p_t\right)_{\Omega_f} + (\nabla r, \nabla(\hat{q}_t - p_t))_{\Omega_f} \\
&\qquad + \rho_f\left(\underset{\sim}{e}_{tt}, \operatorname{div}\left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\right)_{\Omega_s} + \rho_f a\left(\underset{\approx}{E}_{tt}, \left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\right) \\
&\qquad + \left\langle\frac{1}{c}r_t, \hat{q}_t - p_t\right\rangle_{\Gamma_f} + \rho_f\left\langle\left(\rho_s\underset{\approx}{\mathcal{A}_s}\right)^{-1}\underset{\approx}{E}_t\mathbf{n}_s, \left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\mathbf{n}_s\right\rangle_{\Gamma_s} = 0,
\end{aligned}
\tag{3.23}
$$

$$
\begin{aligned}
&\rho_f\left(\rho_s\underset{\sim}{e}_{ttt}, \underset{\sim}{e}_{tt}\right)_{\Omega_s} - \rho_f\left(\operatorname{div}\underset{\approx}{E}_t, \underset{\sim}{e}_{tt}\right)_{\Omega_s} + \rho_f\left(\rho_s\underset{\sim}{e}_{ttt}, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\right)_{\Omega_s} \\
&\qquad\qquad - \rho_f\left(\operatorname{div}\underset{\approx}{E}_t, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\right)_{\Omega_s} = 0.
\end{aligned}
\tag{3.24}
$$

Adding up (3.23) and (3.24) gives

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left\|\frac{1}{c}r_t\right\|^2_{0,\Omega_f} + \|\nabla r\|^2_{0,\Omega_f} + \left\|\sqrt{\rho_f\rho_s}\underset{\sim}{e}_{tt}\right\|^2_{0,\Omega_s} + \rho_f a\left(\underset{\approx}{E}_t, \underset{\approx}{E}_t\right)\right) \\
&\qquad + \left\langle\frac{1}{c}r_t, r_t\right\rangle_{\Gamma_f} + \left\langle\left(\rho_s\underset{\approx}{\mathcal{A}_s}\right)^{-1}\underset{\approx}{E}_t\mathbf{n}_s, \underset{\approx}{E}_t\mathbf{n}_s\right\rangle_{\Gamma_s} \\
&= -\left(\frac{1}{c^2}r_{tt}, \hat{q}_t - p_t\right)_{\Omega_f} - (\nabla r, \nabla(\hat{q}_t - p_t))_{\Omega_f} \\
&\qquad - \rho_f\left(\underset{\sim}{e}_{tt}, \operatorname{div}\left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\right)_{\Omega_s} - \rho_f a\left(\underset{\approx}{E}_{tt}, \underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right) \\
&\qquad - \rho_f\left(\rho_s\underset{\sim}{e}_{ttt}, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\right)_{\Omega_s} + \rho_f\left(\operatorname{div}\underset{\approx}{E}_t, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\right)_{\Omega_s} \\
&\qquad - \left\langle\frac{1}{c}r_t, \hat{q}_t - p_t\right\rangle_{\Gamma_f} - \rho_f\left\langle\left(\rho_s\underset{\approx}{\mathcal{A}_s}\right)^{-1}\underset{\approx}{E}_t\mathbf{n}_s, \left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\mathbf{n}_s\right\rangle_{\Gamma_s} \\
&\leqslant -\left(\frac{1}{c^2}r_{tt}, \hat{q}_t - p_t\right)_{\Omega_f} + \|\nabla r\|^2_{0,\Omega_f} + \|\nabla(\hat{q}_t - p_t)\|^2_{0,\Omega_f} \\
&\qquad - \rho_f\left(\underset{\sim}{e}_{tt}, \operatorname{div}\left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\right)_{\Omega_s} - \rho_f a\left(\underset{\approx}{E}_{tt}, \underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right) \\
&\qquad - \rho_f\left(\rho_s\underset{\sim}{e}_{ttt}, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\right)_{\Omega_s} + \rho_f\left(\operatorname{div}\underset{\approx}{E}_t, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\right)_{\Omega_s} \\
&\qquad + \delta\left\langle\frac{1}{c}r_t, r_t\right\rangle_{\Gamma_f} + C(\delta)\left\langle\hat{q}_t - p_t, \hat{q}_t - p_t\right\rangle_{\Gamma_f} + \delta\left\langle\left(\rho_s\underset{\approx}{\mathcal{A}_s}\right)^{-1}\underset{\approx}{E}_t\mathbf{n}_s, \underset{\approx}{E}_t\mathbf{n}_s\right\rangle_{\Gamma_s} \\
&\qquad + C(\delta)\left\langle\left(\rho_s\underset{\approx}{\mathcal{A}_s}\right)^{-1}\left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\mathbf{n}_s, \left(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\right)\mathbf{n}_s\right\rangle_{\Gamma_s}.
\end{aligned}
\tag{3.25}
$$

From (2.14), (3.13), (3.22) and the fact that div $\underset{\approx}{H}{}_s^h \subset \underset{\sim}{V}{}_s^h$ we obtain that

$$-\rho_f\Big(\rho_s \underset{\sim}{e}_{ttt}, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\Big)_{\Omega_s} + \rho_f\Big(\operatorname{div} \underset{\approx}{E}_t, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\Big)_{\Omega_s} = \Big(\rho_f\Big(Q_h \underset{\sim}{g}_s - \underset{\sim}{g}_s\Big)_t, \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\Big)_{\Omega_s}. \tag{3.26}$$

Let

$$y(t) = \frac{1}{2}\left(\left\|\frac{1}{c^2} r_t\right\|^2_{0,\Omega_f} + \|\nabla r\|^2_{0,\Omega_f} + \left\|\sqrt{\rho_f \rho_s}\,\underset{\sim}{e}_{tt}\right\|^2_{0,\Omega_s}\right) + \frac{1}{2}\rho_f a\Big(\underset{\approx}{E}_t, \underset{\approx}{E}_t\Big),$$

then it follows from (3.25) and (3.26) that

$$\begin{aligned}
&\frac{\mathrm{d}y(t)}{\mathrm{d}t} + \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{E}_t \mathbf{n}_s, \underset{\approx}{E}_t \mathbf{n}_s\Big\rangle_{\Gamma_s} + \left\|\frac{1}{\sqrt{c}} r_t\right\|^2_{0,\Gamma_f} \\
&\quad\leqslant -\left(\frac{1}{c^2} r_{tt}, \hat{q}_t - p_t\right)_{\Omega_f} - \rho_f\Big(\underset{\sim}{e}_{tt}, \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big)\Big)_{\Omega_s} \\
&\qquad - \rho_f a\Big(\underset{\approx}{E}_{tt}, \underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big)_{\Omega_s} + \|\nabla r\|^2_{0,\Omega_f} + C\|\nabla(\hat{q}_t - p_t)\|^2_{0,\Omega_f} \\
&\qquad + C\Big[\|\hat{q}_t - p_t\|^2_{0,\Gamma_f} + \Big\|\Big(Q_h \underset{\sim}{g}_s - \underset{\sim}{g}_s\Big)_t\Big\|^2_{0,\Omega_s} + \Big\|\Big(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big)\mathbf{n}_s\Big\|^2_{0,\Gamma_s} \\
&\qquad + \Big\|\underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\Big\|^2_{0,\Omega_s}\Big].
\end{aligned} \tag{3.27}$$

Integrating (3.27) on $(0, \tau)$ we have

$$\begin{aligned}
&y(\tau) + \int_0^\tau \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{E}_t \mathbf{n}_s, \underset{\approx}{E}_t \mathbf{n}_s\Big\rangle_{\Gamma_s} \mathrm{d}t + \int_0^\tau \left\|\frac{1}{\sqrt{c}} r_t\right\|^2_{0,\Gamma_f} \mathrm{d}t \\
&\quad\leqslant y(0) - \int_0^\tau \Bigg[\left(\frac{1}{c^2} r_{tt}, \hat{q}_t - p_t\right)_{\Omega_f} + \rho_f a\Big(\underset{\approx}{E}_{tt}, \underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big) \\
&\qquad + \rho_f\Big(\underset{\sim}{e}_{tt}, \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big)\Big)_{\Omega_s} - \|\nabla r\|^2_{0,\Omega_f} - C\|\nabla(\hat{q}_t - p_t)\|^2_{0,\Omega_f} \\
&\qquad - C\|\hat{q}_t - p_t\|^2_{0,\Gamma_f} - \Big\|\Big(Q_h \underset{\sim}{g}_s - \underset{\sim}{g}_s\Big)_t\Big\|^2_{0,\Omega_s} - \Big\|\Big(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big)\mathbf{n}_s\Big\|^2_{0,\Gamma_s} \\
&\qquad - \Big\|\underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt}\Big\|^2_{0,\Omega_s}\Bigg]\mathrm{d}t.
\end{aligned} \tag{3.28}$$

Integrating by parts we get

$$\begin{aligned}
&-\int_0^\tau \left(\frac{1}{c^2} r_{tt}, \hat{q}_t - p_t\right)_{\Omega_f} \mathrm{d}t = -\left(\frac{1}{c^2} r_t(\tau), \hat{q}_t(\tau) - p_t(\tau)\right)_{\Omega_f} \\
&\quad + \left(\frac{1}{c^2} r_t(0), \hat{q}_t(0) - p_t(0)\right)_{\Omega_f} + \int_0^\tau \left(\frac{1}{c^2} r_t, \hat{q}_{tt} - p_{tt}\right)_{\Omega_f} \mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \delta \left\| \frac{1}{c} r_t(\tau) \right\|_{0,\Omega_f}^2 + C(\delta) \|\hat{q}_t(\tau) - p_t(\tau)\|_{0,\Omega_f}^2 \\
&+ C \int_0^\tau \left(\|r_t(t)\|_{0,\Omega_f}^2 + \|\hat{q}_{tt}(t) - p_{tt}(t)\|_{0,\Omega_f}^2 \right) \mathrm{d}t \left(\frac{1}{c^2} r_t(0), \hat{q}_t(0) - p_t(0) \right)_{\Omega_f}.
\end{aligned} \tag{3.29}
$$

Similarly,

$$
\begin{aligned}
-\int_0^\tau \rho_f a\Big(\underset{\approx}{E}_{tt}, \underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big) \mathrm{d}t &\leqslant \delta \left\| \frac{1}{c} \underset{\approx}{E}_t(\tau) \right\|_{0,\Omega_s}^2 + C(\delta) \left\| \underset{\approx}{\hat{\chi}}_t(\tau) - \underset{\approx}{\sigma}_t(\tau) \right\|_{0,\Omega_s}^2 \\
&+ \int_0^\tau \left(\left\| \underset{\approx}{E}_t(t) \right\|_{0,\Omega_s}^2 + \left\| \underset{\approx}{\hat{\chi}}_{tt}(t) - \underset{\approx}{\sigma}_{tt}(t) \right\|_{0,\Omega_s}^2 \right) \mathrm{d}t + \rho_f a\Big(\underset{\approx}{E}_t(0), \underset{\approx}{\hat{\chi}}_t(0) - \underset{\approx}{\sigma}_t(0)\Big)_{\Omega_s},
\end{aligned} \tag{3.30}
$$

and

$$
\begin{aligned}
\int_0^\tau \rho_f \Big(\underset{\sim}{e}_{tt}, \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t\Big)\Big)_{\Omega_s} \mathrm{d}t &\leqslant \delta \left\| \frac{1}{c} \underset{\sim}{e}_t(\tau) \right\|_{0,\Omega_f}^2 + C(\delta) \left\| \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}_t(\tau) - \underset{\approx}{\sigma}_t(\tau)\Big) \right\|_{0,\Omega_f}^2 \\
&+ \int_0^\tau \left(\left\| \underset{\sim}{e}_t(t) \right\|_{0,\Omega_f}^2 + \left\| \underset{\approx}{\hat{\chi}}_{tt}(t) - \underset{\approx}{\sigma}_{tt}(t) \right\|_{0,\Omega_f}^2 \right) \mathrm{d}t - \rho_f \Big(\underset{\sim}{e}_t(0), \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}_t(0) - \underset{\approx}{\sigma}_t(0)\Big)\Big)_{\Omega_f}.
\end{aligned} \tag{3.31}
$$

It follows from (3.29)–(3.31) and the following inequality:

$$
\|\phi(\tau)\|_{0,\Omega_f}^2 \leqslant \|\phi(0)\|_{0,\Omega_f}^2 + \int_0^\tau [\|\phi(t)\|_{0,\Omega_f}^2 + \|\phi_t(t)\|_{0,\Omega_f}^2] \mathrm{d}t, \tag{3.32}
$$

that

$$
\begin{aligned}
y(\tau) &+ \int_0^\tau \left[\left\langle \Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{E}_t \mathbf{n}_s, \underset{\approx}{E}_t \mathbf{n}_s \right\rangle_{\Gamma_s} + \left\| \frac{1}{\sqrt{c}} r_t \right\|_{0,\Gamma_f}^2 \right] \mathrm{d}t \\
&\leqslant \delta \Big(\|r_t(\tau)\|_{0,\Omega_f}^2 + \left\| \underset{\approx}{E}_t(\tau) \right\|_{0,\Omega_s}^2 + \left\| \underset{\sim}{e}_t(\tau) \right\|_{0,\Omega_s}^2 \Big) \\
&\quad + \int_0^\tau \Big[y(t) + \|\nabla(\hat{q}_t - p_t)\|_{0,\Omega_f}^2 + \|\hat{q}_{tt} - p_{tt}\|_{0,\Omega_f}^2 + \|\hat{q}_t - p_t\|_{0,\Gamma_f}^2 \\
&\quad + \left\| \underset{\sim}{\hat{v}}_{tt} - \underset{\sim}{u}_{tt} \right\|_{0,\Omega_s}^2 + \left\| \Big(Q_h \underset{\sim}{g}_s - \underset{\sim}{g}_s\Big)_t \right\|_{0,\Omega_s}^2 + \left\| \underset{\approx}{\hat{\chi}}_{tt} - \underset{\approx}{\sigma}_{tt} \right\|_{0,\Omega_s}^2 \\
&\quad + \left\| \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}_{tt} - \underset{\approx}{\sigma}_{tt}\Big) \right\|_{0,\Omega_s}^2 + \left\| \underset{\approx}{\hat{\chi}}_t - \underset{\approx}{\sigma}_t \right\|_{0,\Gamma_s}^2 \Big] \mathrm{d}t + Cy(0).
\end{aligned} \tag{3.33}
$$

Finally, the desired estimate (3.15) follows by taking the constant δ small enough in (3.33) and then applying Gronwall's inequality. The proof is completed. □

We now are ready to state the main theorem of this section.

THEOREM 2 Suppose there exists a constant $C_0 > 0$ which is independent of h such that the starting values satisfy

$$\text{(i)}\quad \left\| \underset{\sim}{U}_1 - \underset{\sim}{u}_1 \right\|_{L^2(\Omega_s)} + \left\| \underset{\sim}{U}_1 - \underset{\sim}{u}_1 \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} + \left\| \underset{\approx}{\Pi}^0 - \underset{\approx}{\sigma}_0 \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} \leqslant C_0\, h^{k+1},$$

$$\text{(ii)}\quad \|P_0 - p_0\|_{H^1(\Omega_f)} + \|P_1 - p_1\|_{H^1(\Omega_f)} + \left\| \underset{\sim}{U}_0 - \underset{\sim}{u}_0 \right\|_{H^1(\Omega_s)} + \left\| \underset{\sim}{U}_1 - \underset{\sim}{u}_1 \right\|_{H^1(\Omega_s)} + \left\| \underset{\approx}{\Pi}^0 - \underset{\approx}{\sigma}_0 \right\|_{H^1(\Omega_s)} \leqslant C_0\, h^k.$$

Also, suppose that

$$C_1 := \|p\|_{H^2(H^{k+1}(\Omega_f))} + \left\| \underset{\sim}{u} \right\|_{H^2(H^k(\Omega_s))} + \left\| \underset{\approx}{\sigma} \right\|_{H^2(H^{k+1}(\Omega_s))} + \left\| \underset{\sim}{g}_s \right\|_{H^1(H^k(\Omega_s))} < \infty.$$

Then, there exists an h-independent constant $C > 0$ such that

$$\begin{aligned} &\|p_t - P_t\|_{L^\infty(L^2(\Omega_f))} + \|\nabla(p - P)\|_{L^\infty(L^2(\Omega_f))} + \|p_t - P_t\|_{L^2(L^2(\Gamma_f))} \\ &+ \left\| \underset{\sim}{u}_{tt} - \underset{\sim}{U}_{tt} \right\|_{L^\infty(L^2(\Omega_s))} + \left\| \underset{\approx}{\sigma}_t - \underset{\approx}{\Pi}_t \right\|_{L^\infty(L^2(\Omega_s))} + \left\| (\underset{\approx}{\sigma}_t - \underset{\approx}{\Pi}_t)\mathbf{n}_s \right\|_{L^2(L^2(\Gamma_s))} \\ &+ \left\| \operatorname{div}\left(\underset{\approx}{\sigma} - \underset{\approx}{\Pi} \right) \right\|_{L^\infty(L^2(\Omega_s))} \leqslant C(C_0 + C_1)h^k. \end{aligned} \tag{3.34}$$

Proof. The proof is based on an application of Lemma 3. To this end, the main task is to bound $\|\underset{\approx}{E}_t(0)\|_{L^2(\Omega_s)}$ and $\|\underset{\sim}{e}_{tt}(0)\|_{L^2(\Omega_s)}$.

We recall that $(p, \underset{\sim}{u}, \underset{\approx}{\sigma})$ and $(P, \underset{\sim}{U}, \underset{\approx}{\Pi})$ are the solutions of the initial value problems (2.12)–(2.16) and (3.11)–(3.14), respectively. Subtracting (3.12) from (2.13) and (3.13) from (2.14) and setting $t = 0$ in the resulting equations gives the following error equations at $t = 0$:

$$a\left(\underset{\approx}{E}_t(0), \underset{\approx}{\chi}_h\right) + \left(\underset{\sim}{e}_t(0), \operatorname{div} \underset{\approx}{\chi}_h\right)_{\Omega_s} + \left\langle \left(\rho_s \underset{\approx}{\mathcal{A}}_s\right)^{-1} \underset{\approx}{E}(0)\mathbf{n}_s, \underset{\approx}{\chi}_h \mathbf{n}_s \right\rangle_{\Gamma_s} - \left\langle \underset{\sim}{e}_t(0), \underset{\approx}{\chi}_h \mathbf{n}_s \right\rangle_{\Gamma} = 0, \tag{3.35}$$

$$\left(\rho_s \underset{\sim}{e}_{tt}(0), \underset{\sim}{v}_h\right)_{\Omega_s} - \left(\operatorname{div} \underset{\approx}{E}(0), \underset{\sim}{v}_h\right)_{\Omega_s} = 0, \tag{3.36}$$

for any $(q_h, \underset{\sim}{v}_h, \underset{\approx}{\chi}_h) \in \mathbf{W}^h$.

Taking $\underset{\sim}{v}_h = \underset{\sim}{e}_{tt}(0) + (Q_h \underset{\sim}{u}_{tt}(0) - \underset{\sim}{u}_{tt}(0))$ in (3.36) and using $\underset{\approx}{E}(0) = \underset{\approx}{\sigma}_0 - \underset{\approx}{\Pi}^0$, it follows from the Schwarz inequality, (A1), and (3.7) that

$$\begin{aligned} \overline{\rho_s} \left\| \underset{\sim}{e}_{tt}(0) \right\|^2_{L^2(\Omega_s)} &\leqslant \left(\rho_s \underset{\sim}{e}_{tt}(0), \underset{\sim}{e}_{tt}(0)\right)_{\Omega_s} \\ &= \left(\operatorname{div} \underset{\approx}{E}(0), \underset{\sim}{e}_{tt}(0)\right)_{\Omega_s} + \left(\operatorname{div} \underset{\approx}{E}(0), Q_h \underset{\sim}{u}_{tt}(0) - \underset{\sim}{u}_{tt}(0)\right) \\ &\quad - \left(\rho_s \underset{\sim}{e}_{tt}(0), Q_h \underset{\sim}{u}_{tt}(0) - \underset{\sim}{u}_{tt}(0)\right) \\ &\leqslant \frac{\overline{\rho_s}}{2} \left\| \underset{\sim}{e}_{tt}(0) \right\|^2_{L^2(\Omega_s)} + C \left\| \operatorname{div}\left(\underset{\approx}{\sigma}_0 - \underset{\approx}{\Pi}^0\right) \right\|^2_{L^2(\Omega_s)} + C h^{2k} \left\| \underset{\sim}{u}_{tt}(0) \right\|^2_{H^k(\Omega_s)}, \end{aligned}$$

which, with the assumption on the initial data, implies that

$$\left\| \underset{\sim}{e}_{tt}(0) \right\|_{L^2(\Omega_s)} \leqslant C\,h^k. \tag{3.37}$$

Next, taking $\underset{\approx}{\chi}_h = \pi_h \underset{\approx}{\sigma}_t(0) - \underset{\approx}{\Pi}_t(0)$ and writing $\underset{\approx}{E}_t(0) = (\underset{\approx}{\sigma}_t(0) - \pi_h \underset{\approx}{\sigma}_t(0)) + \underset{\approx}{\chi}_h$ in (3.35), it then follows from the Schwarz inequality, the trace inequality, and the inverse inequality (bounding H^1-norm in terms of L^2-norm) (see Brenner & Scott, 1994; Ciarlet, 1978; Arnold *et al.*, 1984) that

$$\begin{aligned}
a\Big(\underset{\approx}{\chi}_h, \underset{\approx}{\chi}_h\Big) &= -\Big(\underset{\sim}{e}_t(0), \operatorname{div} \underset{\approx}{\chi}_h\Big)_{\Omega_s} - \Big\langle \Big(\rho_s \underset{\approx}{\mathcal{A}}_s\Big)^{-1} \underset{\approx}{E}(0)\mathbf{n}_s, \underset{\approx}{\chi}_h \mathbf{n}_s \Big\rangle_{\Gamma_s} \\
&\quad + \Big\langle \underset{\sim}{e}_t(0), \underset{\approx}{\chi}_h \mathbf{n}_s \Big\rangle_{\Gamma} - a\left(\underset{\approx}{\sigma}_t(0) - \pi_h \underset{\approx}{\sigma}_t(0), \underset{\approx}{\chi}_h\right) \\
&\leqslant \left\| \underset{\sim}{e}_t(0) \right\|_{L^2(\Omega_s)} \left\| \operatorname{div} \underset{\approx}{\chi}_h \right\|_{L^2(\Omega_s)} + C \left\| \underset{\approx}{E}(0)\mathbf{n}_s \right\|_{H^{-\frac{1}{2}}(\Gamma_s)} \left\| \underset{\approx}{\chi}_h \mathbf{n}_s \right\|_{H^{\frac{1}{2}}(\Gamma_s)} \\
&\quad + \left\| \underset{\sim}{e}_t(0) \right\|_{H^{-\frac{1}{2}}(\Gamma)} \left\| \underset{\approx}{\chi}_h \mathbf{n}_s \right\|_{H^{\frac{1}{2}}(\Gamma)} + C \left\| \underset{\approx}{\sigma}_t(0) - \pi_h \underset{\approx}{\sigma}_t(0) \right\|_{L^2(\Omega_s)} \left\| \underset{\approx}{\chi}_h \right\|_{L^2(\Omega_s)} \\
&\leqslant c_0 \left\| \underset{\approx}{\chi}_h \right\|^2_{L^2(\Omega_s)} + C\,h^{-2} \Big[\left\| \underset{\sim}{e}_t(0) \right\|^2_{L^2(\Omega_s)} + \left\| \underset{\sim}{e}_t(0) \right\|^2_{H^{-\frac{1}{2}}(\Gamma)} \\
&\quad + \left\| \underset{\approx}{E}(0)\mathbf{n}_s \right\|^2_{H^{-\frac{1}{2}}(\Gamma_s)} \Big] + C \left\| \underset{\approx}{\sigma}_t(0) - \pi_h \underset{\approx}{\sigma}_t(0) \right\|^2_{L^2(\Omega_s)}
\end{aligned} \tag{3.38}$$

for some undetermined constant $c_0 > 0$. Noticing that $\underset{\sim}{e}_t(0) = u_1 - U_1$ and $\underset{\approx}{E}(0) = \underset{\approx}{\sigma}_0 - \underset{\approx}{\Pi}^0$, then from (3.38), the coercivity of $a(\cdot, \cdot)$ on $\underset{\approx}{H}_s \times \underset{\approx}{H}_s$ (see Arnold *et al.*, 1984), and the approximation properties (3.4) and (3.5), the assumption on the initial data, and choosing c_0 small enough we obtain

$$\left\| \underset{\approx}{\chi}_h \right\|_{L^2(\Omega_s)} \leqslant C\,h^k \qquad \text{and} \qquad \left\| \underset{\approx}{E}_t(0) \right\|_{L^2(\Omega_s)} \leqslant C\,h^k. \tag{3.39}$$

Next, we write $\underset{\approx}{E} = (\underset{\approx}{\sigma} - \pi_h \underset{\approx}{\sigma}) + (\pi_h \underset{\approx}{\sigma} - \underset{\approx}{\Pi})$, since $\operatorname{div} \underset{\approx}{H}{}_s^h \subset \underset{\sim}{V}{}_s^h$ and

$$\Big(\operatorname{div}\Big(\underset{\approx}{\sigma} - \pi_h \underset{\approx}{\sigma}\Big), \underset{\sim}{v}_h\Big) = 0 \qquad \forall \underset{\sim}{v}_h \in \underset{\sim}{V}{}_s^h,$$

after choosing $\underset{\sim}{v}_h = \operatorname{div}(\pi_h \underset{\approx}{\sigma} - \underset{\approx}{\Pi})$ in error equation (3.22) and using the Schwarz inequality we get

$$\left\| \operatorname{div}\Big(\pi_h \underset{\approx}{\sigma} - \underset{\approx}{\Pi}\Big) \right\|_{L^\infty(L^2(\Omega_s))} \leqslant \underline{\rho_s} \left\| \underset{\sim}{u}_{tt} - \underset{\sim}{U}_{tt} \right\|_{L^\infty(L^2(\Omega_s))}, \tag{3.40}$$

which then gives an upper bound of $\mathrm{O}(h^k)$ for the last term on the left-hand side of (3.34) if $\|\underset{\sim}{u}_{tt} - \underset{\sim}{U}_{tt}\|_{L^\infty(L^2(\Omega_s))}$, which appears on the left-hand side of (3.15), is controlled by a bound of same order.

Finally, the desired estimate (3.34) follows from (3.15), (3.37), (3.39), (3.40), and the approximation properties (3.1)–(3.5). □

REMARK 2 (a) Estimate (3.34) is optimal for $\|p - P\|_{L^\infty(H^1(\Omega_f))}$ and $\|\operatorname{div}(\underset{\approx}{\sigma} - \underset{\approx}{\Pi})\|_{L^\infty(L^2(\Omega_s))}$ but quasi-optimal for $\|p - P\|_{W^{1,\infty}(L^2(\Omega_f))}$, $\|u - U\|_{W^{2,\infty}(L^2(\Omega_s))}$ and $\|\underset{\approx}{\sigma} - \underset{\approx}{\Pi}\|_{W^{1,\infty}(L^2(\Omega_s))}$.

(b) Clearly, Theorem 2 imposes strong constraints on the choices of approximations of the initial data. To weaken the constraints, we have to estimate the error in weaker norms. This can be done by integrating (2.12) and (3.11) in time t, instead of differentiating (2.13) and (3.12) as was done in the beginning of the proof of Lemma 3. In fact, such an idea was used in the proof of Lemma 1. We also note that the assumption (i) is consistent with the assumption (ii) in the light of Lemma 2.3 of Feng (2000).

(c) In the case $k = 1$, a pair of stable mixed finite element subspaces was constructed for the linear elasticity by Johnson & Mercier (1978). Since the Johnson–Mercier element does not satisfy the inclusion $\operatorname{div} \underset{\approx}{H}_s^h \subset \underset{\sim}{V}_s^h$, the conclusion of Theorem 2 may not hold. However, it is possible to show a weaker result by modifying the proofs of Lemma 3 and Theorem 2.

4. Fully discrete finite element approximations

In this section we shall introduce some fully discrete second-order-in-time finite element methods for the initial-boundary value problem (2.1)–(2.9) by discretizing the system of ordinary differential equations (3.11)–(3.14) using the finite difference time-stepping method. We shall derive error estimates which are analogous to those of Theorem 2, and some new estimates in weaker norms.

4.1 *Formulation of fully discrete finite element methods*

Let J be a positive integer. Let $\Delta t = \frac{T}{J}$, $t_n = n\Delta t$, and

$$\underset{\sim}{u}^n = \underset{\sim}{u}(t_n), \qquad p^n = p(t_n), \qquad \underset{\sim}{U}^n = \underset{\sim}{U}(t_n),$$
$$P^n = P(t_n), \qquad \underset{\approx}{\sigma}^n = \underset{\approx}{\sigma}(t_n), \qquad \underset{\approx}{\Pi}^n = \underset{\approx}{\Pi}(t_n).$$

We also let

$$P^{n+\frac{1}{2}} = \frac{P^n + P^{n+1}}{2}, \qquad \partial_f P^n = \frac{P^{n+1} - P^n}{\Delta t},$$
$$\partial_b P^n = \frac{P^n - P^{n-1}}{\Delta t}, \qquad \partial_c P^n = \frac{P^{n+1} - P^{n-1}}{2\Delta t},$$
$$\overline{\partial}_c P^n = \frac{P^{n+1} + P^{n-1}}{2\Delta t}, \qquad \partial^2 P^n = \frac{P^{n+1} - 2P^n + P^{n-1}}{\Delta t^2},$$
$$P^{n,\gamma} = \gamma P^{n-1} + (1 - 2\gamma)P^n + \gamma P^{n+1}.$$

It is easy to check the following identities:

$$\partial_c P^n = \partial_f P^{n-\frac{1}{2}} = \partial_b P^{n+\frac{1}{2}} = \frac{P^{n+\frac{1}{2}} - P^{n-\frac{1}{2}}}{\Delta t}, \qquad \partial^2 P^n = \partial_f \partial_b P^n,$$

$$\partial_f P^{n,\frac{1}{4}} = \partial_c P^{n+\frac{1}{2}} = \frac{\partial_f P^{n+\frac{1}{2}} + \partial_f P^{n-\frac{1}{2}}}{2}, \qquad \partial^2 P^{n+\frac{1}{2}} = \partial_f \partial_c P^n.$$

Our fully discrete finite element method is defined as seeking a sequence $\{(P^n, \underset{\sim}{U}^n, \underset{\approx}{\Pi}^n)\}_{n=0}^J$ in $\mathbf{W}^h$ such that for any $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h}) \in \mathbf{W}^h$

$$\begin{aligned}
&\left(\frac{1}{c^2}\partial^2 P^n, q_h\right)_{\Omega_f} + (\nabla P^{n,\frac{1}{4}}, \nabla q_h)_{\Omega_f} + \left\langle \frac{1}{c}\partial_c P^n, q_h\right\rangle_{\Gamma_f} \\
&\qquad - \left\langle \rho_f \partial^2 \underset{\sim}{U}^n \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma} = (g_f^{n,\frac{1}{4}}, q_h)_{\Omega_f}, \quad n = 1, 2, \ldots, J-1, \qquad (4.1)
\end{aligned}$$

$$\begin{aligned}
&a\Big(\partial_f \underset{\approx}{\Pi}^n, \underset{\approx}{\chi_h}\Big) + \Big(\partial_f \underset{\sim}{U}^n, \operatorname{div} \underset{\approx}{\chi_h}\Big)_{\Omega_s} + \Big\langle \Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{\Pi}^{n+\frac{1}{2}} \mathbf{n}_s, \underset{\approx}{\chi_h} \mathbf{n}_s\Big\rangle_{\Gamma_s} \\
&\qquad - \Big\langle \partial_f \underset{\sim}{U}^n, \underset{\approx}{\chi_h} \mathbf{n}_s\Big\rangle_{\Gamma} = 0, \quad n = 0, 1, 2, \ldots, J-1, \qquad (4.2)
\end{aligned}$$

$$\Big(\rho_s \partial^2 \underset{\sim}{U}^n, \underset{\sim}{v_h}\Big)_{\Omega_s} - \Big(\operatorname{div} \underset{\approx}{\Pi}^{n,\frac{1}{4}}, \underset{\sim}{v_h}\Big)_{\Omega_s} = \Big(\underset{\sim}{g_s}^{n,\frac{1}{4}}, \underset{\sim}{v_h}\Big)_{\Omega_s}, \quad n = 1, 2, \ldots, J-1, \qquad (4.3)$$

$$\Big(P^0, \underset{\sim}{U}^0, \underset{\approx}{\Pi}^0\Big) \in \mathbf{W}^h, \quad \Big(P^1, \underset{\sim}{U}^1\Big) \in \mathbf{W}^h, \qquad (4.4)$$

where $(P^0, \underset{\sim}{U}^0, \underset{\approx}{\Pi}^0)$ and $(P^1, \underset{\sim}{U}^1)$ are some starting values, which will be specified in the next section.

REMARK 3

(a) Notice that (4.1) and (4.3) are two-step schemes, but (4.2) is a one-step scheme. To start the whole algorithm, we first need to set $n = 0$ in (4.2) to generate $\underset{\approx}{\Pi}^1$.

(b) Equation (4.2) can be regarded as a central difference descretization of (3.12) at $t = t_{n+\frac{1}{2}} := \frac{t_n + t_{n+1}}{2}$ with mesh size $\frac{\Delta t}{2}$. As expected, it results in second-order truncation error in Δt.

(c) For each fixed n, (4.1)–(4.4) can be expressed as a square linear system, hence, to verify its well-posedness, it suffices to show the uniqueness. To this end, it is sufficient to show that the trivial solution is the only solution for zero sources and zero initial data. Again, for the readers' convenience, we sketch the proof in the following. Applying $\rho_f \partial_c$ to (4.2) we get

$$\begin{aligned}
\rho_f a\Big(\partial^2 \underset{\approx}{\Pi}^{n+\frac{1}{2}}, \underset{\approx}{\chi_h}\Big) + \rho_f\Big(\partial^2 \underset{\sim}{U}^{n+\frac{1}{2}}, \operatorname{div} \underset{\approx}{\chi_h}\Big)_{\Omega_s} + \rho_f\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \partial_c \underset{\approx}{\Pi}^{n+\frac{1}{2}} \mathbf{n}_s, \underset{\approx}{\chi_h} \mathbf{n}_s\Big\rangle_{\Gamma_s}& \\
- \rho_f \Big\langle \partial^2 \underset{\sim}{U}^{n+\frac{1}{2}}, \underset{\approx}{\chi_h} \mathbf{n}_s\Big\rangle_{\Gamma} = 0.&
\end{aligned}$$

Next, adding the above equation to the one resulting from averaging (4.1) at n and

$n+1$ steps, and applying $\rho_f \partial_f$ to (4.3) yields

$$\left(\frac{1}{c^2}\partial^2 P^{n+\frac{1}{2}}, q_h\right)_{\Omega_f} + (\nabla P^{n+\frac{1}{2},\frac{1}{4}}, \nabla q_h)_{\Omega_f} + \rho_f\left(\partial^2 \underset{\sim}{U}^{n+\frac{1}{2}}, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s}$$
$$+ \rho_f a\left(\partial^2 \underset{\approx}{\Pi}^{n+\frac{1}{2}}, \underset{\approx}{\chi_h}\right) + \left\langle \frac{1}{c}\partial_c P^{n+\frac{1}{2}}, q_h\right\rangle_{\Gamma_f}$$
$$+ \rho_f\left\langle\left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1}\partial_c \underset{\approx}{\Pi}^{n+\frac{1}{2}}\mathbf{n}_s, \underset{\approx}{\chi_h}\mathbf{n}_s\right\rangle_{\Gamma_s} = (g_f^{n+\frac{1}{2},\frac{1}{4}}, q_h)_{\Omega_f}, \tag{4.5}$$

$$\left(\rho_f \rho_s \partial_f \partial^2 \underset{\sim}{U}^n, \underset{\sim}{v_h}\right)_{\Omega_s} - \rho_f\left(\operatorname{div} \partial_c \underset{\approx}{\Pi}^{n+\frac{1}{2}}, \underset{\sim}{v_h}\right)_{\Omega_s} = \rho_f\left(\partial_c \underset{\sim}{g_s}^{n+\frac{1}{2}}, \underset{\sim}{v_h}\right)_{\Omega_s}. \tag{4.6}$$

Now, suppose that $g_f^j = 0$, $\underset{\sim}{g_s}^j = 0$ for $j = 1, 2, \ldots, J$, and $P^0 = P_1 = 0$, $\underset{\sim}{U}^0 = \underset{\sim}{U}^1 = 0$, $\underset{\approx}{\Pi}^0 = 0$ (hence, $\underset{\approx}{\Pi}^1 = 0$). First, set $(q_h, \underset{\approx}{\chi_h}) = (\partial_c P^{n+\frac{1}{2}}, \partial_c \underset{\approx}{\Pi}^{n+\frac{1}{2}})$ in (4.5) and $v_h = \rho_f \partial^2 \underset{\sim}{U}^{n+\frac{1}{2}}$ in (4.6), then add the resulting equations. Finally, applying the discrete Gronwall inequality to the combined equation. Then we get $P^n = 0$, $\underset{\sim}{U}^n = 0$ and $\underset{\approx}{\Pi}^n = 0$ for $0 \leqslant n \leqslant J$. The proof is completed.

4.2 *Error estimates*

For the fully discrete mixed finite element method (4.5)–(4.6), there holds the following error estimate.

THEOREM 3 In addition to the regularity assumption of Theorem 2, suppose

$$\|\partial_f r^{\frac{1}{2}}\|_{0,\Omega_f} + \|\nabla r^{1,\frac{1}{4}}\|_{0,\Omega_f} + \left\|\partial_f \underset{\sim}{e}^{\frac{1}{2}}\right\|_{0,\Omega_s} + \left\|\partial^2 \underset{\sim}{e}^1\right\|_{0,\Omega_s}$$
$$+ a\left(\partial_f \underset{\approx}{E}^{\frac{1}{2}}, \partial_f \underset{\approx}{E}^{\frac{1}{2}}\right)^{\frac{1}{2}} = \mathrm{O}(h^k + (\Delta t)^2), \tag{4.7}$$

then, there exists an h-independent constant $C > 0$ such that

$$\|\partial_f(p-P)\|_{\tilde{L}^\infty(L^2(\Omega_f))} + \|\nabla(p-P)\|_{\hat{L}^\infty(L^2(\Omega_f))} + \left\|\partial^2(\underset{\sim}{u} - \underset{\sim}{U})\right\|_{\tilde{L}^\infty(L^2(\Omega_s))}$$
$$+ \left\|\partial_f\left(\underset{\approx}{\sigma} - \underset{\approx}{\Pi}\right)\right\|_{\tilde{L}^\infty(L^2(\Omega_s))} + \left\|\operatorname{div}\left(\underset{\approx}{\sigma} - \underset{\approx}{\Pi}\right)\right\|_{\tilde{L}^\infty(L^2(\Omega_f))} \leqslant C[h^k + (\Delta t)^2], \tag{4.8}$$

where

$$\|f\|_{\tilde{L}^\infty(X)} = \max_{1 \leqslant l < J} \|f^{l+\frac{1}{2}}\|_X, \qquad \|f\|_{\hat{L}^\infty(X)} = \max_{1 \leqslant l < J} \|f^{l,\frac{1}{4}}\|_X.$$

Proof. Since the proof is analogous to that of Theorem 2, in the following we shall only highlight the steps which are different. First notice that $(p^n, \underset{\sim}{u}^n, \underset{\approx}{\sigma}^n)$ satisfies

$$\left(\frac{1}{c^2}\partial^2 p^{n+\frac{1}{2}}, q_h\right)_{\Omega_f} + (\nabla p^{n+\frac{1}{2},\frac{1}{4}}, \nabla q_h)_{\Omega_f} + \rho_f\left(\partial^2 \underset{\sim}{u}^{n+\frac{1}{2}}, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s}$$

$$
\begin{aligned}
&+\rho_f a\Big(\partial^2 \underset{\approx}{\sigma}^{n+\frac{1}{2}}, \underset{\approx}{\chi_h}\Big) + \Big\langle \frac{1}{c}\partial_c p^{n+\frac{1}{2}}, q_h\Big\rangle_{\Gamma_f} + \rho_f\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1}\partial_c \underset{\approx}{\sigma}^{n+\frac{1}{2}}\mathbf{n}_s, \underset{\approx}{\chi_h}\mathbf{n}_s\Big\rangle_{\Gamma_s} \\
&\quad = \Big(g_f^{n+\frac{1}{2},\frac{1}{4}} + \alpha^{n+\frac{1}{2}}, q_h\Big)_{\Omega_f} + \Big\langle \frac{1}{c}\delta^{n+\frac{1}{2}}, q_h\Big\rangle_{\Gamma_f} + \Big\langle \rho_f \underset{\sim}{\psi}^{n+\frac{1}{2}}\cdot\mathbf{n}_s, q_h\Big\rangle_{\Gamma} \\
&\qquad + \rho_f\Big(\partial_c \underset{\sim}{\beta}^n, \operatorname{div}\underset{\approx}{\chi_h}\Big)_{\Omega_s} + \rho_f a\Big(\partial_c \underset{\approx}{\gamma}^n, \underset{\approx}{\chi_h}\Big) + \rho_f\Big\langle \partial_c \underset{\sim}{\beta}^n, \underset{\approx}{\chi_h}\mathbf{n}_s\Big\rangle_{\Gamma} \qquad (4.9)
\end{aligned}
$$

$$
\Big(\rho_s\partial^2 \underset{\sim}{u}^n, \underset{\sim}{v_h}\Big)_{\Omega_s} - \Big(\operatorname{div}\underset{\approx}{\sigma}^{n,\frac{1}{4}}, \underset{\sim}{v_h}\Big)_{\Omega_s} = \Big(\underset{\sim}{g_s}^{n,\frac{1}{4}} + \underset{\sim}{\psi}^n, \underset{\sim}{v_h}\Big)_{\Omega_s}, \qquad (4.10)
$$

for any $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h}) \in \mathbf{W}^h$, where

$$
\begin{aligned}
&\alpha^n = \partial^2 p^n - p_{tt}^{n,\frac{1}{4}}, \qquad \underset{\sim}{\beta}^n = \partial_f \underset{\sim}{u}^n - \underset{\sim}{u_t}^{n+\frac{1}{2}}, \qquad \underset{\approx}{\gamma}^n = \partial_f \underset{\approx}{\sigma}^n - \underset{\approx}{\sigma_t}^{n+\frac{1}{2}}, \\
&\delta^n = \partial_c p^n - p_t^{n,\frac{1}{4}}, \qquad \underset{\sim}{\psi}^n = \partial^2 \underset{\sim}{u}^n - \underset{\sim}{u_{tt}}^{n,\frac{1}{4}}.
\end{aligned}
$$

It follows from Taylor's formula that

$$
\begin{aligned}
&\|\alpha^n\|^2_{0,\Omega_f} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \|p_{tttt}(t)\|^2_{0,\Omega_f}\,\mathrm{d}t, \qquad \Big\|\underset{\sim}{\beta}^n\Big\|^2_{0,\Omega_f} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \Big\|\underset{\sim}{u_{ttt}}(t)\Big\|^2_{0,\Omega_s}\mathrm{d}t, \\
&\Big\|\underset{\sim}{\beta}^n\Big\|^2_{-\frac{1}{2},\Gamma} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \Big\|\underset{\sim}{u_{ttt}}(t)\Big\|^2_{0,\Omega_s}\mathrm{d}t, \qquad \Big\|\underset{\approx}{\gamma}^n\Big\|^2_{0,\Omega_f} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \Big\|\underset{\approx}{\sigma_{ttt}}(t)\Big\|^2_{0,\Omega_s}\mathrm{d}t, \\
&\|\delta^n\|^2_{-\frac{1}{2},\Gamma_f} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \|p_{ttt}(t)\|^2_{0,\Omega_f}\,\mathrm{d}t, \qquad \Big\|\underset{\sim}{\psi}^n\Big\|^2_{0,\Omega_f} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \Big\|\underset{\sim}{u_{tttt}}(t)\Big\|^2_{0,\Omega_s}\mathrm{d}t, \\
&\Big\|\underset{\sim}{\psi}^n\Big\|^2_{-\frac{1}{2},\Gamma} \leqslant C(\Delta t)^3\int_{t_{n-1}}^{t_{n+1}} \Big\|\underset{\sim}{u_{tttt}}(t)\Big\|^2_{0,\Omega_s}\mathrm{d}t.
\end{aligned}
$$

Set $r^n = p^n - P^n$, $\underset{\sim}{e}^n = \underset{\sim}{u}^n - \underset{\sim}{U}^n$, $\underset{\approx}{E}^n = \underset{\approx}{\sigma}^n - \underset{\approx}{\Pi}^n$. From (4.5)–(4.6) and (4.9)–(4.10) we get for any $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h}) \in \mathbf{W}^h$

$$
\begin{aligned}
&\Big(\frac{1}{c^2}\partial^2 r^{n+\frac{1}{2}}, q_h\Big)_{\Omega_f} + (\nabla r^{n+\frac{1}{2},\frac{1}{4}}, \nabla q_h)_{\Omega_f} + \rho_f\Big(\partial^2 \underset{\sim}{e}^{n+\frac{1}{2}}, \operatorname{div}\underset{\approx}{\chi_h}\Big)_{\Omega_s} \\
&\quad + \rho_f a\Big(\partial^2 \underset{\approx}{E}^{n+\frac{1}{2}}, \underset{\approx}{\chi_h}\Big) + \Big\langle \frac{1}{c}\partial_c r^{n+\frac{1}{2}}, q_h\Big\rangle_{\Gamma_f} + \rho_f\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}\mathbf{n}_s, \underset{\approx}{\chi_h}\mathbf{n}_s\Big\rangle_{\Gamma_s} \\
&\quad = (\alpha^{n+\frac{1}{2}}, q_h)_{\Omega_f} + \Big\langle \frac{1}{c}\delta^{n+\frac{1}{2}}, q_h\Big\rangle_{\Gamma_f} + \Big\langle \rho_f \underset{\sim}{\psi}^{n+\frac{1}{2}}\cdot\mathbf{n}_s, q_h\Big\rangle_{\Gamma} \\
&\qquad + \rho_f\Big(\partial_c \underset{\sim}{\beta}^n, \operatorname{div}\underset{\approx}{\chi_h}\Big)_{\Omega_s} + \rho_f a\Big(\partial_c \underset{\approx}{\gamma}^n, \underset{\approx}{\chi_h}\Big) + \rho_f\Big\langle \partial_c \underset{\sim}{\beta}^n, \underset{\approx}{\chi_h}\mathbf{n}_s\Big\rangle_{\Gamma}, \qquad (4.11)
\end{aligned}
$$

$$
\Big(\rho_s\partial^2 \underset{\sim}{e}^n, \underset{\sim}{v_h}\Big)_{\Omega_s} - \Big(\operatorname{div}\underset{\approx}{E}^{n,\frac{1}{4}}, \underset{\sim}{v_h}\Big)_{\Omega_s} = \Big(\underset{\sim}{\psi}^n, \underset{\sim}{v_h}\Big)_{\Omega_s}. \qquad (4.12)
$$

Applying the operator ∂_f to (4.12) yields

$$
\Big(\rho_s\partial_f\partial^2 \underset{\sim}{e}^n, \underset{\sim}{v_h}\Big)_{\Omega_s} - \Big(\operatorname{div}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}, \underset{\sim}{v_h}\Big)_{\Omega_s} = \Big(\partial_f \underset{\sim}{\psi}^n, \underset{\sim}{v_h}\Big)_{\Omega_s}. \qquad (4.13)
$$

For any $(\hat{q}^n, \hat{\underset{\sim}{v}}^n, \hat{\underset{\approx}{\chi}}^n) \in \mathbf{W}^h$, set $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h})$ in (4.11) and (4.13) as follows:

$$q_h = \partial_c r^{n+\frac{1}{2}} + \partial_c \hat{q}^{n+\frac{1}{2}} - \partial_c p^{n+\frac{1}{2}}, \qquad \underset{\sim}{v_h} = \rho_f \Big(\partial^2 \underset{\sim}{e}^{n+\frac{1}{2}} + \partial^2 \hat{\underset{\sim}{v}}^{n+\frac{1}{2}} - \partial^2 \underset{\sim}{u}^{n+\frac{1}{2}}\Big),$$

$$\underset{\approx}{\chi_h} = \partial_c \underset{\approx}{E}^{n+\frac{1}{2}} + \partial_c \hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \partial_c \underset{\approx}{\sigma}^{n+\frac{1}{2}},$$

and we obtain that

$$\begin{aligned}
&\frac{1}{2\Delta t}\left[\left\|\frac{1}{c}\partial_f r^{n+\frac{1}{2}}\right\|^2_{0,\Omega_f} - \left\|\frac{1}{c}\partial_f r^{n-\frac{1}{2}}\right\|^2_{0,\Omega_f} + \|\nabla r^{n+1,\frac{1}{4}}\|^2_{0,\Omega_f} - \|\nabla r^{n,\frac{1}{4}}\|^2_{0,\Omega_f}\right] \\
&\quad + \frac{1}{2\Delta t}\left[\|\sqrt{\rho_s\rho_f}\,\partial^2 e^{n+1}\|^2_{0,\Omega_s} - \left\|\sqrt{\rho_s\rho_f}\,\partial^2 \underset{\sim}{e}^n\right\|^2_{0,\Omega_s}\right] \\
&\quad + \frac{1}{2\Delta t}\Big[a\Big(\partial_f \underset{\approx}{E}^{n+\frac{1}{2}}, \partial_f \underset{\approx}{E}^{n+\frac{1}{2}}\Big) - a\Big(\partial_f \underset{\approx}{E}^{n-\frac{1}{2}}, \partial_f \underset{\approx}{E}^{n-\frac{1}{2}}\Big)\Big] \\
&\quad + \left\|\frac{1}{\sqrt{c}}\partial_c r^{n+\frac{1}{2}}\right\|^2_{0,\Gamma_f} + \rho_f\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}\mathbf{n}_s, \partial_c \underset{\approx}{E}^{n+\frac{1}{2}}\mathbf{n}_s\Big\rangle_{\Gamma_s} \\
&= -\left(\frac{1}{c^2}\partial^2 r^{n+\frac{1}{2}}, \partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\right)_{\Omega_f} - (\nabla r^{n+\frac{1}{2},\frac{1}{4}}, \nabla\partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}}))_{\Omega_f} \\
&\quad - \rho_f\Big(\partial^2 \underset{\sim}{e}^{n+\frac{1}{2}}, \operatorname{div}\partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} - \rho_f a\Big(\partial^2 \underset{\approx}{E}^{n+\frac{1}{2}}, \partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} \\
&\quad - \rho_f\Big(\rho_s\partial_f\partial^2 \underset{\sim}{e}^n, \partial^2\Big(\hat{\underset{\sim}{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} - \left\langle\frac{1}{c}\partial_c r^{n+\frac{1}{2}}, \partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\right\rangle_{\Gamma_f} \\
&\quad - \rho_f\Big(\operatorname{div}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}, \partial^2\Big(\hat{\underset{\sim}{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} + (\alpha^{n+\frac{1}{2}}, q_h)_{\Omega_f} + \rho_f\Big(\partial_c \underset{\sim}{\beta}^n, \operatorname{div}\underset{\approx}{\chi_h}\Big)_{\Omega_s} \\
&\quad - \rho_f\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}\mathbf{n}_s, \partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\mathbf{n}_s\Big\rangle_{\Gamma_s} + \rho_f a\Big(\partial_c \underset{\approx}{\gamma}^n, \underset{\approx}{\chi_h}\Big)_{\Omega_s} \\
&\quad + \left\langle\frac{1}{c}\delta^{n+\frac{1}{2}}, q_h\right\rangle_{\Gamma_f} + \Big\langle\rho_f \underset{\sim}{\psi}^{n+\frac{1}{2}}\cdot\mathbf{n}_s, q_h\Big\rangle_{\Gamma} + \rho_f\Big\langle\partial_c \underset{\sim}{\beta}^n, \underset{\approx}{\chi_h}\mathbf{n}_s\Big\rangle_{\Gamma} + \rho_f\Big(\partial_f \underset{\sim}{\psi}^n, \underset{\sim}{v_h}\Big)_{\Omega_s} \\
&\leqslant -\left(\frac{1}{c^2}\partial^2 r^{n+\frac{1}{2}}, \partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\right)_{\Omega_f} + \|\nabla r^{n+\frac{1}{2},\frac{1}{4}}\|^2_{0,\Omega_f} + \|\nabla\partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\|^2_{\Omega_f} \\
&\quad - \rho_f\Big(\partial^2 \underset{\sim}{e}^{n+\frac{1}{2}}, \operatorname{div}\partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} - \rho_f a\Big(\partial^2 \underset{\approx}{E}^{n+\frac{1}{2}}, \partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} \\
&\quad - \rho_f\Big(\rho_s\partial_f\partial^2 \underset{\sim}{e}^n, \partial^2\Big(\hat{\underset{\sim}{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} + \delta\left\|\frac{1}{\sqrt{c}}\partial_c r^{n+\frac{1}{2}}\right\|^2_{0,\Gamma_f} \\
&\quad - \rho_f\Big(\operatorname{div}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}, \partial^2\Big(\hat{\underset{\sim}{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} + C(\delta)\|\partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\|^2_{\Gamma_f} \\
&\quad + \delta\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1}\partial_c \underset{\approx}{E}^{n+\frac{1}{2}}\mathbf{n}_s, \partial_c \underset{\approx}{E}^{n+\frac{1}{2}}\mathbf{n}_s\Big\rangle_{\Gamma_s} \\
&\quad + C(\delta)\Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1}\partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\mathbf{n}_s, \partial_c\Big(\hat{\underset{\approx}{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\mathbf{n}_s\Big\rangle_{\Gamma_s}
\end{aligned}$$

$$
\begin{aligned}
&+ C\Big(\|\alpha^{n+\frac{1}{2}}\|_{0,\Omega_f}^2 + \|\partial_c \underset{\sim}{\beta}^n\|_{0,\Omega_s}^2 + \|\partial_c \underset{\approx}{\gamma}^n\|_{0,\Omega_s}^2 \\
&+ \|\delta^{n+\frac{1}{2}}\|_{-\frac{1}{2},\Gamma_f}^2 + \|\underset{\sim}{\psi}^{n+\frac{1}{2}} \cdot \mathbf{n}_s\|_{-\frac{1}{2},\Gamma}^2 + \|\partial_c \underset{\sim}{\beta}^n\|_{-\frac{1}{2},\Gamma}^2 + \|\partial_f \underset{\sim}{\psi}^n\|_{0,\Omega_s}^2 \\
&+ \|\partial_c (r^{n+\frac{1}{2}} + \hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\|_{1,\Omega_f}^2 + \|\partial_c\Big(\underset{\approx}{E}^{n+\frac{1}{2}} + \underset{\approx}{\hat{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\|_{1,\Omega_s}^2 \\
&+ \|\partial^2 \underset{\sim}{e}^{n+\frac{1}{2}} + \partial_c\Big(\underset{\sim}{\hat{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\|_{0,\Omega_s}^2 + \|\partial_c \underset{\approx}{E}^{n+\frac{1}{2}} \mathbf{n}_s\|_{0,\Omega_s}^2 \Big).
\end{aligned} \tag{4.14}
$$

From (2.14), (4.3) and (4.13) we obtain that

$$
\begin{aligned}
&- \rho_f\Big(\rho_s \partial_f \partial^2 \underset{\sim}{e}^n, \partial^2\Big(\underset{\sim}{\hat{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} - \rho_f\Big(\operatorname{div} \partial_c \underset{\approx}{E}^{n+\frac{1}{2}}, \partial^2\Big(\underset{\sim}{\hat{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} \\
&\qquad = \rho_f\Big(\partial_f\Big(Q_h \underset{\sim}{g_s}^{n,\frac{1}{4}} - \underset{\sim}{g_s}^{n,\frac{1}{4}} + \underset{\sim}{\psi}^n\Big), \partial^2\Big(\underset{\sim}{\hat{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} \\
&\qquad = \rho_f\Big(\partial_c\Big(Q_h \underset{\sim}{g_s}^{n+\frac{1}{2}} - \underset{\sim}{g_s}^{n+\frac{1}{2}}\Big) + \partial_f \underset{\sim}{\psi}^n, \partial^2\Big(\underset{\sim}{\hat{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s},
\end{aligned} \tag{4.15}
$$

$$
\Big\|\operatorname{div} \underset{\approx}{E}^{n,\frac{1}{4}}\Big\|_{0,\Omega_s}^2 \leqslant C\Big(\Big\|Q_h \underset{\sim}{g_s}^{n,\frac{1}{4}} - \underset{\sim}{g_s}^{n,\frac{1}{4}}\Big\|_{0,\Omega_s}^2 + \Big\|\partial^2 \underset{\sim}{e}^n\Big\|_{0,\Omega_s}^2\Big). \tag{4.16}
$$

Applying $\Delta t\ \sum_{n=1}^{l}$ ($l \leqslant J-1$) to both sides of (4.14) with δ sufficiently small,

$$
\begin{aligned}
&\|\partial_f r^{l+\frac{1}{2}}\|_{0,\Omega_f}^2 + \|\nabla r^{l+1,\frac{1}{4}}\|_{0,\Omega_f}^2 + \Big\|\sqrt{\rho_s\rho_f}\,\partial^2 \underset{\sim}{e}^{l+1}\Big\|_{0,\Omega_s}^2 \\
&\qquad + a\Big(\partial_f \underset{\approx}{E}^{l+\frac{1}{2}}, \partial_f \underset{\approx}{E}^{l+\frac{1}{2}}\Big) + \Delta t \sum_{n=1}^{l}\Big[\|\partial_c r^{n+\frac{1}{2}}\|_{0,\Gamma_f}^2 + \Big\|(\partial_c \underset{\approx}{E}^{n+\frac{1}{2}} \mathbf{n}_s)\Big\|_{0,\Gamma_s}^2\Big] \\
&\leqslant C\Delta t \sum_{n=1}^{l}\Bigg[-\bigg(\frac{1}{c^2}\partial^2 r^{n+\frac{1}{2}}, \partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\bigg)_{\Omega_f} + \|\nabla r^{n+\frac{1}{2},\frac{1}{4}}\|_{0,\Omega_f}^2 \\
&\qquad + \|\nabla \partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\|_{\Omega_f}^2 - \rho_f\Big(\partial^2 \underset{\sim}{e}^{n+\frac{1}{2}}, \operatorname{div} \partial_c\Big(\underset{\approx}{\hat{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} \\
&\qquad - \rho_f a\Big(\partial^2 \underset{\approx}{E}^{n+\frac{1}{2}}, \partial_c\Big(\underset{\approx}{\hat{\chi}}^{n+\frac{1}{2}} - \underset{\approx}{\sigma}^{n+\frac{1}{2}}\Big)\Big) \\
&\qquad - \rho_f\Big(\partial_c\Big(Q_h \underset{\sim}{g_s}^{n+\frac{1}{2}} - \underset{\sim}{g_s}^{n+\frac{1}{2}}\Big) + \partial_f \underset{\sim}{\psi}^n, \partial^2\Big(\underset{\sim}{\hat{v}}^{n+\frac{1}{2}} - \underset{\sim}{u}^{n+\frac{1}{2}}\Big)\Big)_{\Omega_s} \\
&\qquad + \|\partial_c(\hat{q}^{n+\frac{1}{2}} - p^{n+\frac{1}{2}})\|_{\Gamma_f}^2 + \Big\langle\Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \partial_c \underset{\approx}{E}^{n+\frac{1}{2}} \mathbf{n}_s, \partial_c \underset{\approx}{E}^{n+\frac{1}{2}} \mathbf{n}_s\Big\rangle_{\Gamma_s} \\
&\qquad + \|\alpha^{n+\frac{1}{2}}\|_{0,\Omega_f}^2 + \Big\|\partial_c \underset{\sim}{\beta}^n\Big\|_{0,\Omega_s}^2 + \Big\|\partial_c \underset{\approx}{\gamma}^n\Big\|_{0,\Omega_s}^2 \\
&\qquad + \|\delta^{n+\frac{1}{2}}\|_{-\frac{1}{2},\Gamma_f}^2 + \Big\|\underset{\sim}{\psi}^{n+\frac{1}{2}} \cdot \mathbf{n}_s\Big\|_{-\frac{1}{2},\Gamma}^2 + \Big\|\partial_c \underset{\sim}{\beta}^n\Big\|_{-\frac{1}{2},\Gamma}^2 + \Big\|\partial_f \underset{\sim}{\psi}^n\Big\|_{0,\Omega_s}^2)
\end{aligned}
$$

$$+\|\partial_c(r^{n+\frac12}+\hat q^{n+\frac12}-p^{n+\frac12})\|^2_{1,\Omega_f}+\left\|\partial_c\left(\underset{\approx}{E}^{n+\frac12}+\underset{\approx}{\hat\chi}^{n+\frac12}-\underset{\approx}{\sigma}^{n+\frac12}\right)\right\|^2_{1,\Omega_s}$$
$$+\left\|\partial^2\underset{\sim}{e}^{n+\frac12}+\partial_c\left(\underset{\sim}{\hat v}^{n+\frac12}-\underset{\sim}{u}^{n+\frac12}\right)\right\|^2_{0,\Omega_s}+\left\|\partial_c\underset{\approx}{E}^{n+\frac12}\mathbf{n}_s\right\|^2_{0,\Omega_s}\Bigg]$$
$$+\|\partial_f r^{\frac12}\|^2_{0,\Omega_f}+\|\nabla r^{1,\frac14}\|^2_{0,\Omega_f}+\left\|\rho_s\rho_f\partial^2\underset{\sim}{e}^1\right\|^2_{0,\Omega_s}+a\left(\partial_f\underset{\approx}{E}^{\frac12},\partial_f\underset{\approx}{E}^{\frac12}\right). \quad (4.17)$$

Using summation by parts we get

$$\Delta t\sum_{n=1}^{l}\left(\frac{1}{c^2}\partial^2 r^{n+\frac12},\partial_c(\hat q^{n+\frac12}-p^{n+\frac12})\right)_{\Omega_f}=\frac{1}{c^2}(\partial_f r^{l+\frac12},\partial_c(\hat q^{l+\frac12}-p^{l+\frac12}))_{\Omega_f}$$
$$-\frac{1}{c^2}(\partial_f r^{\frac12},\partial_c(\hat q^{\frac12}-p^{\frac12}))_{\Omega_f}-\frac{\Delta t}{c^2}\sum_{n=1}^{l}(\partial_f r^{n-\frac12},\partial_f\partial_c(\hat q^{n-\frac12}-p^{n-\frac12}))_{\Omega_s}$$
$$\leqslant\delta\|\partial_f r^{l+\frac12}\|^2_{0,\Omega_f}+C(\delta)\,\|\partial_c(\hat q^{l+\frac12}-p^{l+\frac12})\|^2_{0,\Omega_f}-\frac{1}{c^2}(\partial_f r^{\frac12},\partial_c(\hat q^{\frac12}-p^{\frac12}))_{\Omega_f}$$
$$+\frac{\Delta t}{c^2}\left[\sum_{n=1}^{l}\|\partial_f r^{n-\frac12}\|^2_{0,\Omega_f}+\sum_{n=1}^{l}\|\partial_f\partial_c(\hat q^{n+\frac12}-p^{n+\frac12})\|^2_{\Omega_f}\right], \quad (4.18)$$

$$\Delta t\sum_{n=1}^{l}\rho_f\left(\partial^2\underset{\sim}{e}^{n+\frac12},\operatorname{div}\partial_c\left(\underset{\approx}{\hat\chi}^{n+\frac12}-\underset{\approx}{\sigma}^{n+\frac12}\right)\right)_{\Omega_s}$$
$$\leqslant\delta\left\|\partial_f\underset{\sim}{e}^{l+\frac12}\right\|^2_{0,\Omega_f}+C(\delta)\left\|\operatorname{div}\partial_c\left(\underset{\approx}{\hat\chi}^{l+\frac12}-\underset{\approx}{\sigma}^{l+\frac12}\right)\right\|^2_{\Omega_s}$$
$$-\rho_f\left(\partial_f\underset{\sim}{e}^{\frac12},\operatorname{div}\partial_c\left(\underset{\approx}{\hat\chi}^{\frac12}-\underset{\approx}{\sigma}^{\frac12}\right)\right)_{\Omega_s}$$
$$+\rho_f\Delta t\sum_{n=1}^{l}\left[\left\|\partial_f\underset{\sim}{e}^{n-\frac12}\right\|^2_{0,\Omega_s}+\left\|\partial_f\partial_c\operatorname{div}\left(\underset{\approx}{\hat\chi}^{n+\frac12}-\underset{\approx}{\sigma}^{n+\frac12}\right)\right\|_{\Omega_s}\right], \quad (4.19)$$

$$\Delta t\sum_{n=1}^{l}\rho_f a\left(\partial^2\underset{\approx}{E}^{n+\frac12},\partial_c\left(\underset{\approx}{\hat\chi}^{n+\frac12}-\underset{\approx}{\sigma}^{n+\frac12}\right)\right)$$
$$\leqslant\delta\left\|\partial_f\underset{\approx}{E}^{l+\frac12}\right\|^2_{0,\Omega_s}+C(\delta)\left\|\partial_c\left(\underset{\approx}{\hat\chi}^{l+\frac12}-\underset{\approx}{\sigma}^{l+\frac12}\right)\right\|^2_{0,\Omega_s}+\rho_f a\left(\partial_f\underset{\approx}{E}^{\frac12},\partial_c\left(\underset{\approx}{\hat\chi}^{\frac12}-\underset{\approx}{\sigma}^{\frac12}\right)\right)_{\Omega_s}$$
$$+\rho_f\Delta t\sum_{n=1}^{l}\left[\left\|\partial_f\underset{\approx}{E}^{n-\frac12}\right\|^2_{0,\Omega_s}+\left\|\partial_f\partial_c\left(\underset{\approx}{\hat\chi}^{n+\frac12}-\underset{\approx}{\sigma}^{n+\frac12}\right)\right\|_{\Omega_s}\right], \quad (4.20)$$

$$\Delta t\sum_{n=1}^{l}\rho_f\left(\partial_c\left(Q_h\underset{\sim}{g_s}^{n+\frac12}-\underset{\sim}{g_s}^{n+\frac12}\right)+\partial_f\underset{\sim}{\psi}^n,\partial^2\left(\underset{\sim}{\hat v}^{n+\frac12}-\underset{\sim}{u}^{n+\frac12}\right)\right)_{\Omega_s}$$
$$\leqslant\rho_f\Delta t\sum_{n=0}^{l}\left[\left\|\partial_f\underset{\sim}{\psi}^n\right\|^2_{0,\Omega_s}+\left\|\partial_c\left(Q_h\underset{\sim}{g_s}^{n+\frac12}-\underset{\sim}{g_s}^{n+\frac12}\right)\right\|^2_{0,\Omega_s}\right.$$
$$\left.+\left\|\partial^2\left(\underset{\sim}{\hat v}^{n+\frac12}-\underset{\sim}{u}^{n+\frac12}\right)\right\|^2_{0,\Omega_s}\right]. \quad (4.21)$$

Finally, the desired estimate (4.8) follows from (4.14)–(4.21), (3.1)–(3.5) and applying the discrete Gronwall inequality. □

The above theorem imposes a strong constraint on the starting values of the fully discrete scheme. In particular, it requires information about the second iterates of the errors r^2, $\underset{\sim}{e}^2$ and $\underset{\approx}{E}^2$. To weaken the constraint, we have to estimate the errors in weaker norms. This can be done by utilizing the idea used in the proof of Lemma 1 (see item (b) of the remark following the proof of Theorem 2).

Our second main theorem of this section is the following one.

THEOREM 4 Suppose that the starting values of the fully discrete method satisfy

$$\|\partial_f r^0\|_{0,\Omega_f} + \|r^{\frac{1}{2}}\|_{0,\Omega_f} + \left\|\partial_f \underset{\sim}{e}^0\right\|_{0,\Omega_s} + a\left(\underset{\approx}{E}^0, \underset{\approx}{E}^0\right)^{\frac{1}{2}} = \mathrm{O}(h^k + (\Delta t)^2). \tag{4.22}$$

Then, there exists an h-independent constant $C > 0$ such that

$$\max_{1\leqslant l\leqslant J-1}\left\{\|r^{l+\frac{1}{2}}\|_{L^2(\Omega_f)} + \|\nabla R^l\|_{L^2(\Omega_f)} + \left\|\partial_f \underset{\sim}{e}^l\right\|_{L^2(\Omega_s)} + \left\|\underset{\approx}{E}^{l+\frac{1}{2}}\right\|_{L^2(\Omega_s)}\right\} \leqslant C[h^k + (\Delta t)^2], \tag{4.23}$$

where

$$R^0 := 0, \qquad R^l := \Delta t \sum_{n=1}^{l} r^{n,\frac{1}{4}} \quad \text{for } l \geqslant 1.$$

Proof. The proof is similar to that of Theorem 3, hence we only highlight the main differences. In particular, we pay attention to the dependence on the starting values.

First, the error equations for the scheme (4.1)–(4.4) are given by

$$\left(\frac{1}{c^2}\partial^2 r^n, q_h\right)_{\Omega_f} + (\nabla r^{n,\frac{1}{4}}, \nabla q_h)_{\Omega_f} + \left\langle \frac{1}{c}\partial_c r^n, q_h\right\rangle_{\Gamma_f} - \left\langle \rho_f \partial^2 \underset{\sim}{e}^n \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma}$$
$$= \left(\alpha^n, q_h\right)_{\Omega_f} + \left\langle \frac{1}{c}\delta^n, q_h\right\rangle_{\Gamma_f} + \left\langle \rho_f \underset{\sim}{\psi}^n \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma}, \tag{4.24}$$

$$a\left(\partial_f \underset{\approx}{E}^n, \underset{\approx}{\chi_h}\right) + \left(\partial_f \underset{\sim}{e}^n, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s} + \left\langle \left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1} \underset{\approx}{E}^{n+\frac{1}{2}} \mathbf{n}_s, \underset{\approx}{\chi_h} \mathbf{n}_s\right\rangle_{\Gamma_s}$$
$$- \left\langle \partial_f \underset{\sim}{e}^n, \underset{\approx}{\chi_h}\mathbf{n}_s\right\rangle_{\Gamma} = a\left(\underset{\approx}{\gamma}^n, \underset{\approx}{\chi_h}\right) + \left(\underset{\sim}{\beta}^n, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s} + \left\langle \underset{\sim}{\beta}^n, \underset{\approx}{\chi_h}\mathbf{n}_s\right\rangle_{\Gamma}, \tag{4.25}$$

$$\left(\rho_s \partial^2 \underset{\sim}{e}^n, \underset{\sim}{v_h}\right)_{\Omega_s} - \left(\operatorname{div} \underset{\approx}{E}^{n,\frac{1}{4}}, \underset{\sim}{v_h}\right)_{\Omega_s} = \left(\underset{\sim}{\psi}^n, \underset{\sim}{v_h}\right)_{\Omega_s} \tag{4.26}$$

for any $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h}) \in \mathbf{W}^h$. Note that (4.11) is obtained as a combination of (4.24) and (4.25) after applying the operator $\rho\partial_c$ to (4.25), which is equivalent to taking the derivative in t.

Instead of differentiating (4.25) in t, here we take summation $\Delta t \sum_{n=1}^{m}$ ($m \geqslant 1$) on (4.24), which is equivalent to integrating the equation in t, and yields

$$\left(\frac{1}{c^2}\partial_f r^m, q_h\right)_{\Omega_f} + (\nabla R^m, \nabla q_h)_{\Omega_f} + \left\langle \frac{1}{c} r^{m+\frac{1}{2}}, q_h\right\rangle_{\Gamma_f} - \left\langle \rho_f \partial_f \underset{\sim}{e}^m \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma}$$
$$= \mathcal{S}_1^m + \left(\frac{1}{c^2}\partial_f r^0, q_h\right)_{\Omega_f} + \left\langle \frac{1}{c} r^{\frac{1}{2}}, q_h\right\rangle_{\Gamma_f} + \left\langle \rho_f \partial_f \underset{\sim}{e}^0 \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma}, \quad (4.27)$$

where

$$\mathcal{S}_1^m := \Delta t \sum_{n=1}^{m} \left[(\alpha^n, q_h)_{\Omega_f} + \left\langle \frac{1}{c}\delta^n, q_h\right\rangle_{\Gamma_f} + \left\langle \rho_f \underset{\sim}{\psi}^n \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma} \right].$$

Now, averaging (4.25) and (4.27) at m and $m-1$, respectively, and adding the resulting equations we get

$$\left(\frac{1}{c^2}\partial_f r^{m-\frac{1}{2}}, q_h\right)_{\Omega_f} + (\nabla R^{m-\frac{1}{2}}, \nabla q_h)_{\Omega_f} + \left\langle \frac{1}{c} r^{m,\frac{1}{4}}, q_h\right\rangle_{\Gamma_f} + \rho_f a\left(\partial_f \underset{\approx}{E}^{m-\frac{1}{2}}, \underset{\approx}{\chi_h}\right)$$
$$+ \rho_f\left(\partial_f \underset{\sim}{e}^{m-\frac{1}{2}}, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s} + \rho_f\left\langle \left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1} \underset{\approx}{E}^{m,\frac{1}{4}} \mathbf{n}_s, \underset{\approx}{\chi_h}\mathbf{n}_s\right\rangle_{\Gamma_s}$$
$$= \mathcal{S}_1^{m-\frac{1}{2}} + \mathcal{S}_2^{m-\frac{1}{2}} + \left(\frac{1}{c^2}\partial_f r^0, q_h\right)_{\Omega_f} + \left\langle \frac{1}{c} r^{\frac{1}{2}}, q_h\right\rangle_{\Gamma_f} + \left\langle \rho_f \partial_f \underset{\sim}{e}^0 \cdot \mathbf{n}_s, q_h\right\rangle_{\Gamma}, \quad (4.28)$$

where

$$\mathcal{S}_2^m := \rho_f a\left(\underset{\approx}{\gamma}^m, \underset{\approx}{\chi_h}\right) + \rho_f\left(\underset{\sim}{\beta}^m, \operatorname{div} \underset{\approx}{\chi_h}\right)_{\Omega_s} + \rho_f\left\langle \underset{\sim}{\beta}^m, \underset{\approx}{\chi_h}\mathbf{n}_s\right\rangle_{\Gamma}.$$

For any $(\hat{q}^n, \underset{\sim}{\hat{v}}^n, \underset{\approx}{\hat{\chi}}^n) \in \mathbf{W}^h$, set $(q_h, \underset{\sim}{v_h}, \underset{\approx}{\chi_h})$ in (4.26) and (4.28) as follows:

$$q_h = r^{m,\frac{1}{4}} + \hat{q}^{m,\frac{1}{4}} - p^{m,\frac{1}{4}}, \qquad \underset{\sim}{v_h} = \rho_f\left(\partial_f \underset{\sim}{e}^{m-\frac{1}{2}} + \partial_f \underset{\sim}{\hat{v}}^{m-\frac{1}{2}} - \partial_f \underset{\sim}{u}^{m-\frac{1}{2}}\right),$$
$$\underset{\approx}{\chi_h} = \underset{\approx}{E}^{m,\frac{1}{4}} + \underset{\approx}{\hat{\chi}}^{m,\frac{1}{4}} - \underset{\approx}{\sigma}^{m,\frac{1}{4}},$$

and add the resulting equations to get

$$\frac{1}{2\Delta t}\left[\left\|\frac{1}{c} r^{n+\frac{1}{2}}\right\|_{0,\Omega_f}^2 - \left\|\frac{1}{c} r^{n-\frac{1}{2}}\right\|_{0,\Omega_f}^2, + \|\nabla R^m\|_{0,\Omega_f}^2 - \|\nabla R^{m-1}\|_{0,\Omega_f}^2\right]$$
$$+ \frac{1}{2\Delta t}\left[\|\sqrt{\rho_s\rho_f}\partial_f e^m\|_{0,\Omega_s}^2 - \left\|\sqrt{\rho_s\rho_f}\partial_f \underset{\sim}{e}^{m-1}\right\|_{0,\Omega_s}^2\right]$$
$$+ \frac{1}{2\Delta t}\left[a\left(\underset{\approx}{E}^{m+\frac{1}{2}}, \underset{\approx}{E}^{m+\frac{1}{2}}\right) - a\left(\underset{\approx}{E}^{m-\frac{1}{2}}, \underset{\approx}{E}^{m-\frac{1}{2}}\right)\right]$$
$$+ \left\|\frac{1}{\sqrt{c}} r^{m,\frac{1}{4}}\right\|_{0,\Gamma_f}^2 + \rho_f\left\langle \left(\rho_s \underset{\approx}{\mathcal{A}_s}\right)^{-1} \underset{\approx}{E}^{m,\frac{1}{4}} \mathbf{n}_s, \underset{\approx}{E}^{m,\frac{1}{4}} \mathbf{n}_s\right\rangle_{\Gamma_s}$$
$$= -\left(\frac{1}{c^2}\partial_f r^{m-\frac{1}{2}}, \hat{q}^{m,\frac{1}{4}} - p^{m,\frac{1}{4}}\right)_{\Omega_f} - (\nabla R^{m-\frac{1}{2}}, \nabla \hat{q}^{m,\frac{1}{4}} - p^{m,\frac{1}{4}})_{\Omega_f}$$

$$
\begin{aligned}
&- \rho_f \Big(\partial_f \underset{\sim}{e}^{m-\frac{1}{2}}, \operatorname{div}\Big(\underset{\approx}{\hat{\chi}}^{m,\frac{1}{4}} - \underset{\approx}{\sigma}^{m,\frac{1}{4}}\Big)\Big)_{\Omega_s} - \rho_f a\Big(\partial_f \underset{\approx}{E}^{m-\frac{1}{2}}, \underset{\approx}{\hat{\chi}}^{m,\frac{1}{4}} - \underset{\approx}{\sigma}^{m,\frac{1}{4}}\Big)_{\Omega_s} \\
&- \rho_f \Big(\rho_s \partial^2 \underset{\sim}{e}^{n}, \partial_f \Big(\underset{\sim}{\hat{v}}^{m-\frac{1}{2}} - \underset{\sim}{u}^{m-\frac{1}{2}}\Big)\Big)_{\Omega_s} - \Big\langle \frac{1}{c} r^{m,\frac{1}{4}}, \hat{q}^{m,\frac{1}{4}} - p^{m,\frac{1}{4}}) \Big\rangle_{\Gamma_f} \\
&- \rho_f \Big(\operatorname{div} \underset{\approx}{E}^{m,\frac{1}{4}}, \partial_f \Big(\underset{\sim}{\hat{v}}^{m-\frac{1}{2}} - \underset{\sim}{u}^{m-\frac{1}{2}}\Big)\Big)_{\Omega_s} - \rho_f \Big\langle \Big(\rho_s \underset{\approx}{\mathcal{A}_s}\Big)^{-1} \underset{\approx}{E}^{m,\frac{1}{4}} \mathbf{n}_s, \underset{\approx}{\hat{\chi}}^{m,\frac{1}{4}} - \underset{\approx}{\sigma}^{m,\frac{1}{4}}) \mathbf{n}_s \Big\rangle_{\Gamma_s} \\
&+ \mathcal{S}_1^{m-\frac{1}{2}} + \mathcal{S}_2^{m-\frac{1}{2}} + \Big(\frac{1}{c^2} \partial_f r^0, q_h\Big)_{\Omega_f} + \Big\langle \frac{1}{c} r^{\frac{1}{2}}, q_h \Big\rangle_{\Gamma_f} + \Big\langle \rho_f \partial_f \underset{\sim}{e}^0 \cdot \mathbf{n}_s, q_h \Big\rangle_{\Gamma}.
\end{aligned}
\tag{4.29}
$$

Here we have used the identities

$$
\partial_f R^{m-1} = r^{m,\frac{1}{4}} = \frac{r^{m+\frac{1}{2}} + r^{m-\frac{1}{2}}}{2}, \qquad \partial^2 \underset{\sim}{e}^m = \frac{\partial_f \underset{\sim}{e}^m - \partial_f \underset{\sim}{e}^{m-1}}{2}.
$$

It follows from the error equation (4.25) with $n = 0$ and (4.22) that

$$
a\Big(\underset{\approx}{E}^1, \underset{\approx}{E}^1\Big) \leqslant C[h^k + (\Delta t)^2],
$$

hence

$$
a\Big(\underset{\approx}{E}^{\frac{1}{2}}, \underset{\approx}{E}^{\frac{1}{2}}\Big) \leqslant C[h^k + (\Delta t)^2],
$$

and all starting errors appearing in (4.29) are of $\mathrm{O}(h^k + (\Delta t)^2)$. Since the remaining proof is the same as that of Theorem 3, we omit it. □

REMARK 4 Several choices of $P^0, P^1, \underset{\sim}{U}^0, \underset{\sim}{U}^1$ and $\underset{\approx}{\Pi}^0$, which satisfy the constraint (4.22), are known in the literature: see Cowser *et al.* (1996), Dupont (1973), Feng (1998) and Santos *et al.* (1988) for detailed discussions.

5. Concluding notes

We make two comments about the heterogeneous finite element methods introduced in the previous sections. First, we comment on the construction of the finite element space $\mathbf{W}^h$. Second, we propose some parallelizable iterative solution methods for effectively solving the finite element systems (4.1)–(4.4).

5.1 *Construction of* $\mathbf{W}^h$

We recall that the space $\mathbf{W}^h$ is defined as

$$
\mathbf{W}^h = \Big\{ \Big(q_h, \underset{\sim}{v_h}, \underset{\approx}{\tau_h}\Big) \in P_f^h \times \underset{\sim}{V_s^h} \times \underset{\approx}{H_s^h};\ \underset{\approx}{\tau_h} \mathbf{n}_s - q_h \mathbf{n}_f = 0 \text{ pointwise on } \Gamma \Big\}.
$$

Notice that the constraint in the above definition is exactly the interface condition (2.4). Because this condition is an *essential* boundary condition for the stress tensor $\underset{\approx}{\sigma}$, it cannot be absorbed into the mixed weak formulation (2.12)–(2.16). Although it can be imposed weakly in one way or another, in this paper we choose to (strongly) build it into the finite element spaces because (i) it can be easily satisfied by the finite element spaces P_f^h and $\underset{\approx}{H}{}_s^h$ used in this paper; (ii) it allows us to avoid some technicalities and simplifies the error analysis. In the following we shall explain (i) briefly.

From Arnold *et al.* (1984) we know that for any 'triangle' $K \in \mathcal{T}_h|_{\Omega_s}$ a function $\underset{\approx}{\tau_h} \in \underset{\approx}{H}{}_s^h$ is a piecewise polynomial of degree k ($k \geqslant 1$) on K and $\underset{\approx}{\tau_h}\mathbf{n}_K$ is continuous across the boundary ∂K of the triangle K. Hence, $\underset{\approx}{\tau_h}\mathbf{n}_K|_{\partial K}$ is defined uniquely and its components are polynomials of degree k on each edge of K, in particular, on the edge $e = \partial K \cap \Gamma$ if $\text{meas}(\partial K \cap \Gamma) > 0$. Now let $K' \in \mathcal{T}_h|_{\Omega_f}$ be the 'triangle' which shares the edge $e = \partial K \cap \Gamma$ with the triangle $K \in \mathcal{T}_h|_{\Omega_s}$. For any $q_h \in P_f^h$, this is a polynomial of degree k on K' and is continuous across its boundary $\partial K'$. Clearly, q_h is also a polynomial of degree k on $e = \partial K \cap \Gamma = \partial K' \cap \Gamma$. Hence, for each $1 \leqslant i \leqslant N$, if the nodal parameters of $(\underset{\approx}{\tau_h}\mathbf{n}_K)_i$ (which are indeed used as the degrees of freedom, see p. 15 of Arnold *et al.*, 1984) and the nodal parameters of $(q_h\mathbf{n}_f)_i$ are chosen to be same on each edge $e \subset \Gamma$, then we have $\underset{\approx}{\tau_h}\mathbf{n}_s - q_h\mathbf{n}_f \equiv 0$ on e, therefore, on the whole Γ.

5.2 *Domain decomposition algorithms*

Due to the heterogeneous nature of the fluid–solid interaction problem, it seems that the only practical way for solving the fully discrete finite element system (4.1)–(4.4) is to decouple the problem on the interface and to solve it in the fluid and solid regions separately (in parallel). Such an approach is known as the domain decomposition (divide-and-conquer) approach. Clearly, the crux of the approach is how to do the decoupling and how to piece the subdomain solutions together to get a good whole domain solution. In this section, we propose some domain decomposition algorithms to meet the goal. The algorithms to be introduced are adapted versions of those developed in Feng (2000) for the standard Galerkin approximations of the same fluid–solid interaction problem. For the sake of clarity, we shall present the algorithms at the differential level. Furthermore, no convergence analysis for the proposed algorithms will be given here since a similar analysis can be found in Feng (2000, 1998).

First, from Feng (2000) we know that the interface conditions (2.3) and (2.4) can be equivalently formulated as

$$\frac{\partial p}{\partial \mathbf{n}_f} + \alpha p_t = \rho_f \underset{\sim}{u_{tt}} \cdot \mathbf{n}_s - \alpha\sigma\left(\underset{\sim}{u_t}\right)\mathbf{n}_s \cdot n_s \quad \text{on } \Gamma \times (0, T), \tag{5.1}$$

$$\beta \underset{\approx}{\sigma}\left(\underset{\sim}{u_t}\right)\mathbf{n}_s \cdot \mathbf{n}_s + \rho_f \underset{\sim}{u_{tt}} \cdot \mathbf{n}_s = -\beta p_t + \frac{\partial p}{\partial \mathbf{n}_f} \quad \text{on } \Gamma \times (0, T), \tag{5.2}$$

$$\sigma\left(\underset{\sim}{u_t}\right)\mathbf{n}_s \cdot \tau_s = 0 \quad \text{on } \Gamma \times (0, T), \tag{5.3}$$

for any pair of constants α and β such that $|\alpha| + |\beta| \neq 0$ and $\alpha + \beta \neq 0$ when $\alpha\beta \neq 0$, where τ_s stands for the unit tangent vector on $\partial\Omega_s$.

We remark that to ensure the convergence of the domain decomposition algorithms to be given next, we need to restrict $\alpha > 0$ and $\beta > 0$. However, how to choose α and β and what are the best choices of them remain as open problems.

Based on the above new interface conditions, we propose the following two iterative algorithms. The first resembles the block-Jacobi-type iteration and the second resembles the block Gauss–Seidel-type iteration.

Algorithm 1

Step 1. $\forall (p^0, \underset{\sim}{u}^0, \underset{\approx}{\sigma}^0) \in P_f \times \underset{\sim}{V_s} \times \underset{\approx}{H_s}$.

Step 2. Generate $\{(p^j, \underset{\sim}{u}^j, \underset{\approx}{\sigma}^j)\}_{j\geqslant 1}$ iteratively by solving

$$\frac{1}{c^2} p^j_{tt} - \Delta p^j = g_f \qquad \text{in } \Omega_f \times (0,T), \tag{5.4}$$

$$\frac{1}{c} p^j_t + \frac{\partial p^j}{\partial \mathbf{n}_f} = 0 \qquad \text{on } \Gamma_f \times (0,T), \tag{5.5}$$

$$\frac{\partial p^j}{\partial \mathbf{n}_f} + \alpha p^j_t = \rho_f \underset{\sim}{u}^{j-1}_{tt} \cdot \mathbf{n}_s - \alpha \underset{\approx}{\sigma_t}^{j-1} \mathbf{n}_s \cdot \mathbf{n}_s \qquad \text{on } \Gamma \times (0,T); \tag{5.6}$$

$$\rho_s \underset{\sim}{u}^j_{tt} - \operatorname{div} \underset{\approx}{\sigma}^j = \underset{\sim}{g_s} \qquad \text{in } \Omega_s \times (0,T), \tag{5.7}$$

$$\frac{1}{2\mu_s} \underset{\approx}{\sigma}^j - \gamma_s \operatorname{tr}\left(\underset{\approx}{\sigma}^j\right) - \underset{\approx}{\varepsilon}\left(\underset{\sim}{u}^j\right) = 0 \qquad \text{in } \Omega_s \times (0,T), \tag{5.8}$$

$$\rho_s \underset{\approx}{\mathcal{A}_s} \underset{\sim}{u}^j_t + \underset{\approx}{\sigma}^j \mathbf{n}_s = 0 \qquad \text{on } \Gamma_s \times (0,T), \tag{5.9}$$

$$\beta \underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \mathbf{n}_s + \rho_f \underset{\sim}{u}^j_{tt} \cdot \mathbf{n}_s = -\beta p^{j-1}_t + \frac{\partial p^{j-1}}{\partial \mathbf{n}_f} \qquad \text{on } \Gamma \times (0,T), \tag{5.10}$$

$$\underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \tau_s = 0 \qquad \text{on } \Gamma \times (0,T). \tag{5.11}$$

Algorithm 2

Step 1. $\forall p^0 \in P_f$.

Step 2. Generate $\{(p^j, \underset{\sim}{u}^j, \underset{\approx}{\sigma}^j)\}_{j\geqslant 0}$ iteratively by solving

$$\rho_s \underset{\sim}{u}^j_{tt} - \operatorname{div} \underset{\approx}{\sigma}^j = \underset{\sim}{g_s} \qquad \text{in } \Omega_s \times (0,T), \tag{5.12}$$

$$\frac{1}{2\mu_s} \underset{\approx}{\sigma}^j - \gamma_s \operatorname{tr}\left(\underset{\approx}{\sigma}^j\right) - \underset{\approx}{\varepsilon}\left(\underset{\sim}{u}^j\right) = 0 \qquad \text{in } \Omega_s \times (0,T), \tag{5.13}$$

$$\rho_s \underset{\approx}{\mathcal{A}_s} \underset{\sim}{u}^j_t + \underset{\approx}{\sigma}^j \mathbf{n}_s = 0 \qquad \text{on } \Gamma_s \times (0,T), \tag{5.14}$$

$$\beta \underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \mathbf{n}_s + \rho_f \underset{\sim}{u}^j_{tt} \cdot \mathbf{n}_s = -\beta p^j_t + \frac{\partial p^j}{\partial \mathbf{n}_f} \qquad \text{on } \Gamma \times (0,T), \tag{5.15}$$

$$\underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \tau_s = 0 \qquad \text{on } \Gamma \times (0,T); \tag{5.16}$$

$$\frac{1}{c^2} p^{j+1}_{tt} - \Delta p^{j+1} = g_f \qquad \text{in } \Omega_f \times (0,T), \tag{5.17}$$

$$\frac{1}{c}p_t^{j+1} + \frac{\partial p^{j+1}}{\partial \mathbf{n}_f} = 0 \qquad \text{on } \Gamma_f \times (0,T), \tag{5.18}$$

$$\frac{\partial p^{j+1}}{\partial \mathbf{n}_f} + \alpha p_t^{j+1} = \rho_f \underset{\sim}{u}_{tt}^j \cdot \mathbf{n}_s - \alpha \underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \mathbf{n}_s \quad \text{on } \Gamma \times (0,T). \tag{5.19}$$

REMARK 5 Appropriate initial conditions must be provided in the above algorithms. We omit these conditions for notation brevity. We also point out that the superscript j in the above algorithms stands for the jth iteration, while it was also used in Section 4 to indicate the function value at time step t_j. We assume no ambiguity will be caused by the notation abuse.

Algorithms 1 and 2 require evaluation of the normal derivative of the previous iterates of p on the interface Γ. To avoid the inconvenience and possible loss of accuracy at discrete level, we propose the following modification to the algorithms by introducing a 'Lagrange multiplier'. Let

$$G_s = -\beta p_t + \frac{\partial p}{\partial \mathbf{n}_f} \qquad \text{on } \Gamma.$$

It is easy to check that

$$G_s = -(\alpha+\beta)p_t + \left(\alpha p_t + \frac{\partial p}{\partial \mathbf{n}_f}\right) = -(\alpha+\beta)p_t + \rho_f \underset{\sim}{u}_{tt} \cdot \mathbf{n}_s - \alpha \underset{\approx}{\sigma_t} \mathbf{n}_s \cdot \mathbf{n}_s. \tag{5.20}$$

Based on (5.20) we replace (5.10) in Algorithm 1 by the following updates:

$$\beta \underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \mathbf{n}_s + \rho_f \underset{\sim}{u}_{tt}^j \cdot \mathbf{n}_s = G_s^{j-1} \qquad \text{on } \Gamma \times (0,T), \tag{5.21}$$

$$G_s^j = -(\alpha+\beta)p_t^j + \rho_f \underset{\sim}{u}_{tt}^j \cdot \mathbf{n}_s - \alpha \underset{\approx}{\sigma_t}^j \mathbf{n}_s \cdot \mathbf{n}_s \qquad \text{on } \Gamma \times (0,T), \tag{5.22}$$

and *Step* 1 is replaced by $\forall (G_s^0, \underset{\sim}{u}^0, \underset{\approx}{\sigma}^0) \in L^2(\Gamma) \times \underset{\sim}{V_s} \times \underset{\approx}{H_s}$. Clearly, (5.21) and (5.22) do not involve explicit computation of the normal derivative of p on Γ. Instead, they require one additional function update for G_s. Since the modification for Algorithm 2 is similar, we omit it.

We conclude with two remarks. First, using the energy method of Feng (1998, 2000), it can be shown that the above algorithms all converge for $\alpha > 0$ and $\beta > 0$ strongly in the energy norm of the flow–solid interaction problem (see, Theorem 1). Second, to implement the above algorithms on computers, one has to discretize them first. Different discretization methods can be used in the fluid and solid subdomain. For example, the fully discrete mixed finite element method developed in Section 4 can be employed to do the discretization. After an approximate solution is generated at time step t_j, to compute an approximate solution at the next time step t_{j+1} using the algorithms, it is natural and efficient to use the computed solution at the time step t_j as the initial guess to start the algorithms.

Acknowledgement

The second author was partially supported by the National Science Foundation of China grants 40145020 and 40275023. The authors would like to thank the referees for carefully reading the paper and for their valuable suggestions.

REFERENCES

ARNOLD, D., BREZZI, F. & DOUGLAS JR., J. (1984) PEERS: a new mixed finite element for plate elasticity. *Japan. J. Appl. Math.*, **1**, 1–22.

ARNOLD, D., DOUGLAS JR., J. & GUPTA, C. P. (1984) A family of higher order finite element methods for plate elasticity. *Numer. Math.*, **45**, 1–22.

BABUŠKA, I. (1973) The finite element method with Lagrangian multipliers. *Numer. Math.*, **20**, 179–192.

BRENNER, S. C. & SCOTT, L. R. (1994) *The Mathematical Theory of Finite Element Methods*. New York: Springer.

BREZZI, F. & FORTIN, M. (1991) *Mixed and Hybrid Finite Element Methods*. New York: Springer.

CIARLET, P. (1978) *The Finite Element Method for Elliptic Problems*. New York: North-Holland.

COWSER, L. C., DUPONT, T. F. & WHEELER, M. (1996) A priori estimates for a mixed finite element approximation of second-order hyperbolic equations with absorbing boundary conditions. *SIAM J. Numer. Anal.*, **33**, 492–504.

DEMKOWICZ, L., ODEN, J. T., AINSWORTH, M. & GENG, P. (1991) Solution of elastic scattering problems in linear acoustic using h-p boundary element methods. *J. Comput. Appl. Math.*, **36**, 29–63.

DUPONT, T. (1973) L^2-estimates for Galerkin methods for second order hyperbolic equations. *SIAM J. Numer. Anal.*, **10**, 880–889.

ENGQUIST, B. & MAJDA, A. (1979) Radiation boundary conditions for acoustic and elastic wave calculations. *Commun. Pure Appl. Math.*, **132**, 313–557.

FENG, X. (1998) Analysis of a domain decomposition method for the nearly elastic wave equations based on mixed finite element methods. *IMA J. Numer. Anal.*, **18**, 229–250.

FENG, X. (2000) Analysis of finite element methods and domain decomposition algorithms for a fluid–solid interaction problem. *SIAM J. Numer. Anal.*, **38**, 1312–1336.

FENG, X., LEE, P. & WEI, Y. (2001) Mathematical analysis of a fluid–solid interaction problem. *Appl. Anal.*, **80**, 409–429.

HALE, J. (1969) *Ordinary Differential Equations*. New York: Wiley-Interscience.

JOHNSON, C. & MERCIER, B. (1978) Some equilibrium finite element methods for two-dimensional elasticity problem. *Numer. Math.*, **30**, 103–116.

MAKRIDAKIS, CH. (1992) On mixed finite element methods for linear elastodynamics. *Numer. Math.*, **61**, 235–260.

MAKRIDAKIS, CH., IHLENBURG, F. & BABUŠKA, I. (1996) Analysis and finite element methods for a fluid-solid interaction problem in one dimension. *Math. Models Methods Appl. Sci.*, **6**, 1119–1141.

LYSMER, J. & KUHLMEYER, R. L. (1969) Finite dynamic model for infinite media. *J. Engrg. Mech. Div., Proc. ASCE 95*, **EM4**, 859–877.

QUARTERONI, A., PASQUARELLI, F. & VALLI, A. (1992) Heterogeneous domain decomposition: principles, algorithms and applications. *Proc. 5th Int. Symp. on Domain Decomposition Method for Partial Differential Equations* (D. Keyes *et al.*, eds). Philadelphia, PA: SIAM, pp. 129–150.

SANTOS, J. E., DOUGLAS, JR., J. & CALDERON, A. P. (1988) Finite element methods for a composite model in elastodynamics. *SIAM J. Numer. Anal.*, **25**, 513–532.

WHEELER, M. F. (1973) A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal.*, **10**, 723–759.