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A REDUCED FE FORMULATION BASED ON POD METHOD FOR HYPERBOLIC EQUATIONS*

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Abstract A proper orthogonal decomposition (POD) method was successfully used in the reduced-order modeling of complex systems. In this paper, we extend the applications of POD method, namely, apply POD method to a classical finite element (FE) formulation for second-order hyperbolic equations with real practical applied background, establish a reduced FE formulation with lower dimensions and high enough accuracy, and provide the error estimates between the reduced FE solutions and the classical FE solutions and the implementation of algorithm for solving reduced FE formulation so as to provide scientific theoretic basis for service applications. Some numerical examples illustrate the fact that the results of numerical computation are consistent with theoretical conclusions. Moreover, it is shown that the reduced FE formulation based on POD method is feasible and efficient for solving FE formulation for second-order hyperbolic equations.

Key words proper orthogonal decomposition; finite element formulation; error estimate; hyperbolic equations

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1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with piecewise smooth boundary $\partial\Omega$ and consider the following initial boundary value problem for a second-order hyperbolic equation in $\Omega \times [0, T]$.

Problem I Find u such that

$$\begin{cases} u_{tt} - \varepsilon \triangle u = f(x, y, t), \quad (x, y, t) \in \Omega \times (0, T], \\ u(x, y, t) = 0, \quad (x, y, t) \in \partial\Omega \times (0, T], \\ u(x, y, 0) = \varphi_0(x, y), \quad u_t(x, y, 0) = \varphi_1(x, y), \quad (x, y) \in \Omega, \end{cases} \tag{1.1}$$

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where u_{tt} denotes $\partial^2 u/\partial t^2$, ε is a positive constant, source term $f(x,y,t)$ and initial value functions $\varphi_0(x,y)$ and $\varphi_1(x,y)$ are all smooth enough to ensure the analysis validity, and T is the total time. For the sake of convenience and without loss of generality, we may as well suppose that $\varphi_0(x,y)$ and $\varphi_1(x,y)$ are all zero functions in the following theoretical analysis.

Problem I is used to describe the wave phenomena in the nature such as hydrodynamics, displacement problems in porous media and vibrations of a membrane, acoustic vibrations of a gas, electromagnetic processes in nonconducting. Therefore, it has very important real practical applied background, but usually includes complex computing domain, initial value functions, and source term which is dependent on real practical system. Generally speaking, it is not easy to find their exact solutions for the practical second-order hyperbolic equations; on the contrary, it is an efficient approach to find their numerical solutions (see [1–5]).

The finite element (FE) method is regarded as one of the most effective numerical methods for finding their numerical solutions of second-order hyperbolic equations. However, some classical FE formulations for second-order hyperbolic equations include too many degrees of freedom. Thus, an important problem is how to reduce their degrees of freedom and alleviate the computational load as well as to save time for calculations and resource demands in the practical computational process. This is done in a way that guarantees sufficiently accurate numerical solutions.

It was shown that a proper orthogonal decomposition (POD) method by combining with some numerical methods for partial differential equations can provide efficient means of generating reduced order models and alleviating the computational load and memory requirements (see [6]). POD method was widely and successfully applied to numerous fields, including signal analysis and pattern recognition (see [7]), statistics (see [8]), geophysical fluid dynamics or meteorology (also see [8]). POD method essentially provides an orthogonal basis for representing the given data in a certain least squares optimal sense, that is, it provides a way to find optimal lower dimensional approximations of the given data.

In early time, POD method was mainly used to perform principal component analysis in computations of statistics and search the main behavior of a dynamic system (see [6–8] and their cited references), until the method of snapshots was introduced by Sirovich (see [9]) and was then widely applied for reducing the order of the POD eigenvalue problem. Until ten years ago, some Galerkin POD methods for parabolic problems and a general equation in fluid dynamics were not presented (see [10, 11]). More recently, some reduced order finite difference models and finite element (or mixed finite element) formulations and error estimates for the non-stationary conduction–convection problems, the upper tropical Pacific Ocean model, the non-stationary Navier-Stokes equations, Burgers equations, and parabolic equations based on POD method were presented by our research group (see [12–20]).

To the best of our knowledge, there are no published results addressing the case that a combination of POD method with FE method is used to deal with second-order hyperbolic equations or providing error estimates between classical FE solutions and reduced FE solutions or the implementation of algorithm for solving reduced FE formulation. In this paper, we extend the developments in [10–20], i.e., combine the classical FE method with POD method to establish a reduced FE formulation with lower dimensions and sufficiently high accuracy for second-order hyperbolic equations with real practical applied background and provide the error estimates between the reduced FE solutions and the classical FE solutions and the implementation of algorithm for solving reduced FE formulation so as to provide scientific theoretic basis and computational ways for service applications. Some numerical examples illustrate the fact that the results of numerical computation are consistent with theoretical conclusions. Moreover, it is shown that the reduced FE formulation based on POD method is feasible and efficient for solving second-order hyperbolic equations.

The rest of this paper is organized as follows. Section 2 recalls the classical fully discrete FE formulation for second-order hyperbolic equations and generates snapshots from the first fewer several transient solutions computed from the equation system derived by the classical fully discrete FE formulation. In Section 3, the optimal orthonormal POD bases are reconstructed from the elements of the snapshots with POD method and a reduced fully discrete FE formulation with lower dimensions and sufficiently high accuracy based on POD method for second-order hyperbolic equations is developed. In Section 4, the error estimates between the classical FE solutions and the reduced FE approximate solutions based on POD method for second-order hyperbolic equations and the implementation of algorithm for solving reduced FE formulation are provided. In Section 5, some numerical examples are presented for illustrating that the errors between the reduced FE approximate solutions and the classical FE solutions are consistent with previously obtained theoretical results, thus validating the feasibility and efficiency of POD formulation. Section 6 provides main conclusions and future tentative ideas.

2 Recall Classical Fully Discrete FE Formulation for Problem I and Generation of Snapshots

Sobolev spaces and their norms used in this context are standard (see [21]). Let $H = H_0^1(\Omega)$. Then, the variational formulation for Problem I can be written as follows.

Problem II Find $u(t) : [0,T] \rightarrow H$ such that

$$\begin{cases} (u_{tt}, v) + \varepsilon(\nabla u, \nabla v) = (f, v), & \forall v \in H, \\ u(x,y,0) = 0, \quad u_t(x,y,0) = 0, & (x,y) \in \Omega, \end{cases} \tag{2.1}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(\Omega)$.

Let N be a positive integer, $\tau = T/N$ denote the time step increment. For any function $g(x,y,t)$, we define $g^n = g(x,y,t_n)$ at time $t_n = n\tau$ $(0 \leqslant n \leqslant N)$ and write

$$g^{n,\frac{1}{2}} = \frac{g^{n+1}+g^{n-1}}{2},\ \partial_t g^n = \frac{g^{n+1}-g^n}{\tau},\ \partial_t^2 g^n = \frac{g^{n+1}-2g^n+g^{n-1}}{\tau^2}.$$

Then Problem II has the following equivalent formulation.

Problem III Find $u^{n+1} \in H$ such that

$$\begin{cases} (\partial_t^2 u^n, v) + \varepsilon(\nabla u^{n,\frac{1}{2}}, \nabla v) = (f^{n,\frac{1}{2}}, v) + (R_1^n, \tau), & \forall v \in H, \\ u^0 = 0, \quad u^1 = 0, & (x,y) \in \Omega, \end{cases} \tag{2.2}$$

where $R_1^n = \partial_t^2 u^n - u_{tt} = O(\tau^2 \partial^4 u/\partial t^4)$. If $\varphi_0(x,y)$ and $\varphi_1(x,y)$ are non-zero functions, it is necessary to define $u^0 = \varphi_0(x,y)$, $u^1 = u^0 + 2\tau\varphi_1(x,y) + R_3^n$ $(u^0 = u^{-1}, R_3^n = O(\tau^3\partial^3 u/\partial t^3))$.

Since the left hand side of the first equation in Problem III is a symmetric positive definite and bounded bilinear function on H, it has a unique solution $u^{n+1} \in H$ $(n = 0, 1, \cdots, N-1)$.

Let $\{\Im_h\}$ be a uniformly regular family of triangulation of $\bar{\Omega}$ (see [22–24]). The finite element space is taken as

$$H_h = \{u_h \in H;\ u_h|_K \in P_m(K),\ \forall K \in \Im_h\},$$

where $m \geqslant 1$ and $P_m(K)$ is the space of polynomials of degree $\leqslant m$ on K. If the fully discrete approximation of u is denoted by u_h^n $(n = 1, 2, \cdots, N)$, then, the fully discrete FE formulation for Problem II may be written as

Problem IV Find $u_h^{n+1} \in H_h$ such that

$$\begin{cases} (\partial_t^2 u_h^n, v_h) + \varepsilon(\nabla u_h^{n,\frac{1}{2}}, \nabla v_h) = (f^{n,\frac{1}{2}}, v_h),\ \forall v_h \in H_h,\ n = 1, 2, \cdots, N-1, \\ u_h^0 = 0, \quad u_h^1 = 0. \end{cases} \tag{2.3}$$

If $\varphi_0(x, y)$ and $\varphi_1(x, y)$ are non-zero functions, it is necessary to define $u_h^0 = P_h\varphi_0(x, y)$ and $u_h^1 = u_h^0 + 2\tau P_h\varphi_1(x, y)$ $(u_h^0 = u_h^{-1})$, where $P_h : L^2(\Omega) \to H_h$ is L^2 projection (see [22]). Since the coefficient matrix of equation in Problem IV is symmetric positive definite on H_h, it has a unique set of solutions $u_h^{n+1} \in H_h$ $(n = 1, 2, \cdots, N+1)$ under $u_h^0 = 0$, and $u_h^1 = 0$ and the following theorem of error estimates (see [1]) holds.

Theorem 1 If $f(x, y, t) \in H^2(0, T; H^1(\Omega)) \cap L^2(0, T; H^{m+1}(\Omega))$, then there are the following error estimates between the solution u of Problem II and the solutions u_h^n of Problem IV:

$$\|u(t_n) - u_h^n\|_0 \leqslant C(h^{m+1} + \tau^2), \quad n = 1, 2, \cdots, N, \tag{2.4}$$

where C is a constant independent of h and τ, but dependent on other data ε and f of Problem II.

Thus, if only $f(x, y, t)$, ε, $\varphi_0(x, y)$, $\varphi_1(x, y)$, the triangulation parameter h, the time step increment τ, and the finite element space H_h are given, we can obtain a set of solutions $u_h^n \in H_h$, $n = 1, 2, \cdots, N$, by solving Problem IV. And then we choose the first L (in general, $L^2 = O(N)$; for example, $L = 20$, $N = 200$) instantaneous solutions $u_h^i(x, y)$, $1 \leqslant i \leqslant L$ (which are useful and of interest for us) from N instantaneous solutions $u_h^n(x, y)$, $n = 1, 2, \cdots, N$, for Problem IV, which are referred to as snapshots of POD method.

Remark 1 When one computes actual problems, he may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation). For example, computing for real practical physical systems, one can use theirs previous prediction results to construct the ensemble of snapshots, then reconstruct the POD optimal basis for the ensemble of snapshots by using the following POD method, and finally the finite element space H_h is substituted with the subspace generated with POD basis in order to derive their reduced order physical systems with lower dimensions. Thus, the future change of physical systems can be quickly simulated, which is a result of major importance for real-life applications.

3 Generation of POD Basis and Reduced FE Formulation Based on POD Method for Problem I

For $u_h^i(x,y)$ $(n=1,2,\cdots,L)$ in Section 2, let $W_i(x,y)=u_h^i(x,y)$ $(1\leqslant i\leqslant L)$ and

$$\mathcal{V}=\mathrm{span}\{W_1,W_2,\cdots,W_L\}, \tag{3.1}$$

and refer to $\mathcal{V}$ as the space generated by the snapshots $\{W_i\}_{i=1}^L$ at least one of which is assumed to be non-zero function. Let $\{\psi_j\}_{j=1}^l$ denote an orthonormal basis of $\mathcal{V}$ with dimension $l=\dim\mathcal{V}$. Then each member of the ensemble $\{W_i\}_{i=1}^L$ can be expressed as

$$W_i=\sum_{j=1}^l(W_i,\psi_j)_H\psi_j,\quad i=1,2,\cdots,L, \tag{3.2}$$

where $(W_i,\psi_j)_H=(\nabla u_h^i,\nabla\psi_j)$, $(\cdot,\cdot)$ being the L^2-inner product.

Definition 1 The method of POD consists in finding the orthonormal basis ψ_j $(j=1,2,\cdots,l)$ such that, for every d $(1\leqslant d\leqslant l)$, the mean square error between the elements W_i $(1\leqslant i\leqslant L)$ and corresponding dth partial sum of (3.2) is minimized on average

$$\min_{\{\psi_j\}_{j=1}^d}\frac{1}{L}\sum_{i=1}^L\left\|W_i-\sum_{j=1}^d(W_i,\psi_j)_H\psi_j\right\|_H^2 \tag{3.3}$$

subject to

$$(\psi_r,\psi_j)_H=\delta_{rj},\quad 1\leqslant r\leqslant d, 1\leqslant j\leqslant r, \tag{3.4}$$

where $\|W_i\|_H^2=\|\nabla u_h^i\|_0^2$. A solution $\{\psi_j\}_{j=1}^d$ of (3.3)–(3.4) is known as a POD basis of rank d.

We introduce the Gramian matrix $\boldsymbol{A}=(A_{ij})_{L\times L}\in R^{L\times L}$ corresponding to the snapshots $\{W_i\}_{i=1}^L$ by

$$A_{ij}=\frac{1}{L}(W_i,W_j)_H. \tag{3.5}$$

The matrix $\boldsymbol{A}$ is positive semi-definite and of rank l. Thus, the solution of (3.3)–(3.4) can be found; moreover, there hold the following results (see [9–12]).

Proposition 2 Let $\lambda_1\geqslant\lambda_2\geqslant\cdots\geqslant\lambda_l>0$ denote the positive eigenvalues of $\boldsymbol{A}$ and $\boldsymbol{v}^1$, $\boldsymbol{v}^2$, $\cdots$, $\boldsymbol{v}^l$ the associated orthonormal eigenvectors. Then a POD basis of rank $d\leqslant l$ is given by

$$\psi_i=\frac{1}{\sqrt{L\lambda_i}}\sum_{j=1}^L(\boldsymbol{v}^i)_jW_j,\qquad 1\leqslant i\leqslant d\leqslant l, \tag{3.6}$$

where $(\boldsymbol{v}^i)_j$ denote the j-th component of the eigenvector $\boldsymbol{v}^i$. Furthermore, the following error formula holds

$$\frac{1}{L}\sum_{i=1}^L\left\|W_i-\sum_{j=1}^d(W_i,\psi_j)_H\psi_j\right\|_H^2=\sum_{j=d+1}^l\lambda_j. \tag{3.7}$$

Let $H^d=\mathrm{span}\{\psi_1,\psi_2,\cdots,\psi_d\}$, then $H^d\subset H_h$. For $u\in H_h$, define the Ritz projection P^d: $H_h\to H^d$ by

$$(\nabla P^du,\nabla v_d)=(\nabla u,\nabla v_d),\qquad\forall v_d\in H^d. \tag{3.8}$$

Then there is an extension P^h: $H \to H_h$ of P^d such that $P^h|_{H_h} = P^d : H_h \to H^d$ defined by (see [26])

$$(\nabla P^h u, \nabla v_h) = (\nabla u, \nabla v_h), \quad \forall v_h \in H_h. \tag{3.9}$$

Due to (3.9) the operator P^h is well-defined and bounded (see [18, 20]):

$$\|\nabla P^h u\|_0^2 \leqslant \|\nabla u\|_0^2, \quad \forall u \in H, \tag{3.10}$$

$$\|u - P^d u\|_0 \leqslant Ch\|\nabla(u - P^h u)\|_0, \quad \forall u \in H, \tag{3.11}$$

and there hold the following results (also see [18, 20]).

Lemma 3 For every d $(1 \leqslant d \leqslant l)$, if $\tau = O(h)$, the projection operator P^d satisfies

$$\frac{1}{L}\sum_{i=1}^{L}[\|u_h^i - P^d u_h^i\|_0^2 + h^2\|\nabla(u_h^i - P^d u_h^i)\|_0^2] \leqslant Ch^2 \sum_{j=d+1}^{l} \lambda_j, \tag{3.12}$$

where $u_h^i \in H_h$ is the solution of Problem V.

Thus, by using H^d, we can obtain the reduced fully discrete FE formulation based on POD method for Problem IV as follows.

Problem V Find $u_d^{n+1} \in H^d$ such that

$$\begin{cases} (\partial_t^2 u_d^n, v_d) + \varepsilon(\nabla u_d^{n,\frac{1}{2}}, \nabla v_d) = (f^{n,\frac{1}{2}}, v_d), \ \forall v_d \in H^d, \ n = 1, 2, \cdots, N-1, \\ u_d^0 = 0, \quad u_d^1 = 0. \end{cases} \tag{3.13}$$

If $\varphi_0(x,y)$ and $\varphi_1(x,y)$ are non-zero functions, it is necessary to define $u_d^0 = P^d u_h^0$ and $u_d^1 = u_d^0 + 2\tau P^d \varphi_1(x,y)$ $(u_d^0 = u_d^{-1})$. Since the coefficient matrix of Problem V is symmetric positive definite on H^d, it has a unique set of solutions $u_d^{n+1} \in H^d$ $(n = 1, 2, \cdots, N-1)$ under $u_d^0 = 0$ and $u_d^1 = 0$. Moreover, it is easy to prove that u_d^{n+1} satisfy the following inequality

$$\|u_d^{n+1}\|_0^2 \leqslant C\left(\|u_d^1\|_0^2 + \|u_d^0\|_0^2 + \tau\sum_{i=1}^{n}\|f^{i,\frac{1}{2}}\|_0^2\right)\exp(8\varepsilon T) \ (n = 1, 2, \cdots, N-1), \tag{3.14}$$

which show that the solution of Problem V is stable and continuously dependent on source term $f(x,y,t)$ and initial conditions $\varphi_0(x,y)$ and $\varphi_1(x,y)$ if we don't assume that they are zero functions.

Remark 2 If $\Im_h$ is a uniformly regular triangulation and H_h is taken as the space of piecewise linear functions, the total degrees of freedom for Problem IV, i.e., the number of unknown quantities, is N_h (where N_h is the number of inner vertices of triangulation $\Im_h$, see [22–24]), while the number of total degrees of freedom for Problem V is d $(d \ll l \leqslant L)$. For scientific engineering problems, the number of inner vertices of triangulation $\Im_h$ is more than ten thousands or even more than a hundred million, while d is only the number of few maximal eigenvalues which are chosen to be the first L snapshots from the N solutions, so that it is very small (for example, in Section 5, $d = 6$, while $N_h = 2000 \times 2000 = 4 \times 10^6$). Therefore, Problem V is a reduced fully discrete FE formulation based on POD method for Problem IV. Moreover, since the development and change of many physical systems are closely related to previous results, one may truly capture laws of change of physical systems by using existing results as snapshots to construct POD basis and solve PDEs corresponding to physical systems. Therefore, the POD method provides useful and important application.

4 Error Analysis of Solutions for Problem V and Implementation of Its Algorithm

4.1 Error Estimates of Solutions for Problem V

In this subsection, we recur to the classical FE method to derive the error estimates of solutions for Problem V. To this end, it is necessary to introduce the following discrete Gronwall Lemma (see [22, 25]).

Lemma 4 (Discrete Gronwall Lemma) If $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ are three positive sequences, and $\{c_n\}$ is monotone, satisfying

$$a_n + b_n \leqslant c_n + \bar{\lambda}\sum_{i=0}^{n-1} a_i, \quad \bar{\lambda} > 0, \quad a_0 + b_0 \leqslant c_0,$$

then

$$a_n + b_n \leqslant c_n \exp(n\bar{\lambda}), \quad n \geqslant 0.$$

We have the following main results for Problem V.

Theorem 5 Under hypotheses of Theorem 1, if $k = O(h)$, $L^2 = O(N)$, then the following error estimates hold:

$$\|u_h^n - u_d^n\|_0 \leqslant C\left(\tau \sum_{j=d+1}^{l} \lambda_j\right)^{\frac{1}{2}}, \ n = 1, 2, \cdots, L. \tag{4.1}$$

Proof Since $H^d \subset H_h$, subtracting Problem V from Problem IV taking $v_h = v_d \in H^d$ yields that

$$(\partial_t^2(u_h^n - u_d^n), v_d) + \varepsilon(\nabla(u_h^{n,\frac{1}{2}} - u_d^{n,\frac{1}{2}}), \nabla v_d) = 0, \quad \forall v_d \in H^d, \tag{4.2}$$

Let $\theta = u_h - u_d$. Thus, on the one hand, we have that

$$\begin{aligned}
&(\partial_t^2\theta^n, \theta^{n+1} - \theta^{n-1}) + \frac{\varepsilon}{2}(\nabla(\theta^{n+1} + \theta^{n-1}), \nabla(\theta^{n+1} - \theta^{n-1}))\\
=\ &(\partial_t\theta^n - \partial_t\theta^{n-1}, \partial_t\theta^n + \partial_t\theta^{n-1}) + \frac{\varepsilon}{2}(\nabla(\theta^{n+1} + \theta^{n-1}), \nabla(\theta^{n+1} - \theta^{n-1}))\\
=\ &\|\partial_t\theta^n\|_0^2 - \|\partial_t\theta^{n-1}\|_0^2 + \frac{\varepsilon}{2}\|\nabla\theta^{n+1}\|_0^2 - \frac{\varepsilon}{2}\|\nabla\theta^{n-1}\|_0^2,
\end{aligned} \tag{4.3}$$

and on the other hand, by using Hölder inequality and Cauchy inequality, we have, from (4.2), (3.8), and (3.11), that

$$\begin{aligned}
&(\partial_t^2\theta^n, \theta^{n+1} - \theta^{n-1}) + \frac{\varepsilon}{2}(\nabla(\theta^{n+1} + \theta^{n-1}), \nabla(\theta^{n+1} - \theta^{n-1}))\\
=\ &(\partial_t^2\theta^n, u_h^{n+1} - P^d u^{n+1} - (u_h^{n-1} - P^d u^{n-1})) + (\partial_t^2\theta^n, P^d u_h^{n+1} - u_d^{n+1} - (P^d u_h^{n-1} - u_d^{n-1}))\\
&+\frac{\varepsilon}{2}(\nabla(u_h^{n+1} + u_h^{n-1}) - \nabla P^d(u_h^{n+1} + u_h^{n-1}), \nabla(u_h^{n+1} + u_h^{n-1}) - \nabla P^d(u_h^{n+1} + u_h^{n-1}))\\
&+\frac{\varepsilon}{2}(\nabla(\theta^{n+1} + \theta^{n+1}), \nabla(P^d u_h^{n+1} - u_d^{n+1} - (P^d u_h^{n-1} - u_d^{n-1})))\\
=\ &(\partial_t(u_h^n - u_d^n) - \partial_t(u_h^{n-1} - u_d^{n-1}), \partial_t(u_h^n - P^d u^n) + \partial_t(u_h^{n-1} - P^d u^{n-1}))\\
&+\frac{\varepsilon}{2}(\nabla u_h^{n+1} - \nabla P^d u_h^{n+1} + \nabla u_h^{n-1} - \nabla P^d u_h^{n-1}, \nabla u_h^{n+1} - \nabla P^d u_h^{n+1} - (\nabla u_h^{n-1} - \nabla P^d u_h^{n-1}))\\
\leqslant\ &\tau^{\frac{1}{2}}(\|\partial_t\theta^n\|_0^2 + \|\partial_t\theta^{n-1}\|_0^2) + C\tau^{-\frac{1}{2}}(\|\partial_t(u_h^n - P^d u^n)\|_0^2 + \|\partial_t(u_h^{n-1} - P^d u^{n-1})\|_0^2)\\
&+\frac{\varepsilon}{2}(\|\nabla u_h^{n+1} - \nabla P^d u_h^{n+1}\|_0^2 - \|\nabla u_h^{n-1} - \nabla P^d u_h^{n-1}\|_0^2).
\end{aligned} \tag{4.4}$$

Combining (4.3) with (4.4) yields that

$$\begin{aligned}&\|\partial_t\theta^n\|_0^2-\|\partial_t\theta^{n-1}\|_0^2+\frac{\varepsilon}{2}\|\nabla\theta^{n+1}\|_0^2-\frac{\varepsilon}{2}\|\nabla\theta^{n-1}\|_0^2\\ =&\,C\tau^{-\frac12}(\|\partial_t(u_h^n-P^du^n)\|_0^2+\|\partial_t(u_h^{n-1}-P^du^{n-1})\|_0^2)+\tau^{\frac12}(\|\partial_t\theta^n\|_0^2\\ &+\|\partial_t\theta^{n-1}\|_0^2)+\frac{\varepsilon}{2}(\|\nabla u_h^{n+1}-\nabla P^du_h^{n+1}\|_0^2-\|\nabla u_h^{n-1}-\nabla P^du_h^{n-1}\|_0^2).\end{aligned}\tag{4.5}$$

Noting that $\theta^0=0$ and summing (4.5) from 1 to $n-1$ yield that

$$\|\partial_t\theta^{n-1}\|_0^2+\frac{\varepsilon}{2}\|\nabla\theta^n\|_0^2\leqslant C\tau^{-\frac52}\sum_{i=1}^{n}\|u_h^i-P^du^i\|_0^2+2\tau^{\frac12}\sum_{i=1}^{n-1}\|\partial_t\theta^i\|_0^2+\frac{\varepsilon}{2}\|\nabla u_h^n-\nabla P^du_h^n\|_0^2.\tag{4.6}$$

If τ is sufficiently small, for example, $\tau^{\frac12}\leqslant 1/4$, we obtain that

$$\|\partial_t\theta^{n-1}\|_0^2+\varepsilon\|\nabla\theta^n\|_0^2\leqslant C\tau^{-\frac52}\sum_{i=1}^{n}\|u_h^i-P^du^i\|_0^2+4\tau^{\frac12}\sum_{i=1}^{n-2}\|\partial_t\theta^i\|_0^2+\varepsilon\|\nabla u_h^n-\nabla P^du_h^n\|_0^2.\tag{4.7}$$

By applying Lemma 4 (discrete Gronwall lemma) to (4.7), if $\tau=O(h)$ and $L^2=O(N)$ (i.e., $\tau^{\frac12}=O(L^{-1})$), we have from (4.7) and Lemma 3, that

$$\begin{aligned}\|\partial_t\theta^{n-1}\|_0^2+\varepsilon\|\nabla\theta^n\|_0^2&\leqslant\left[C\tau^{-\frac52}\sum_{i=1}^{n}\|u_h^i-P^du^i\|_0^2+\varepsilon\|\nabla u_h^n-\nabla P^du_h^n\|_0^2\right]\exp(4L\tau^{\frac12})\\ &\leqslant C\|\nabla u_h^n-\nabla P^du_h^n\|_0^2+C\tau^{-\frac52}\sum_{i=1}^{n}\|u_h^i-P^du_h^i\|_0^2\\ &\leqslant C\tau^{-\frac12}\sum_{j=d+1}^{l}\lambda_j.\end{aligned}\tag{4.8}$$

Moreover, by extracting the square root of (4.8), and then using triangular inequality, finally squaring it, we get from Lemma 3 that

$$\|u_h^n-u_d^n\|_0^2\leqslant\|u_h^{n-1}-u_d^{n-1}\|_0^2+C\tau^{\frac32}\sum_{j=d+1}^{l}\lambda_j.\tag{4.9}$$

Note that $u_h^0=u_d^0=0$. Summing (4.9) from 1 to n yields that

$$\|u_h^n-u_d^n\|_0^2\leqslant CL\tau^{\frac32}\sum_{j=d+1}^{l}\lambda_j\leqslant C\tau\sum_{j=d+1}^{l}\lambda_j,\tag{4.10}$$

which yields (4.1). □

Combining Theorem 1 with Theorem 5 yields the following result.

Theorem 6 Under hypotheses of Theorem 5, the error estimates between the solutions for Problem II and the solutions for the reduced Problem V are

$$\|u(t_n)-u_d^n\|_0\leqslant C\tau^2+Ch^{m+1}+C\bigg(\tau\sum_{j=d+1}^{l}\lambda_j\bigg)^{\frac12},\quad n=1,2,\cdots,L.\tag{4.11}$$

Remark 3 Theorems 5 and 6 provide the error estimates between the solutions of the reduced FE formulation Problem V and the solutions of classical FE formulation Problem IV and Problem II, respectively. Our method here employs some FE solutions u_h^n for Problem IV as assistant analysis. However, when one computes actual hyperbolic equations, he may obtain the ensemble of snapshots from physical system trajectories by drawing samples from experiments and interpolation (or data assimilation) or previous results. Therefore, the assistant u_h^n could be replaced by the interpolation functions of experimental and previous results, thus rendering it unnecessary to solve Problem IV, and requiring only to solve reduced Problem V directly such that Theorem 5 is satisfied. And then, time t_L is continuously extrapolated forward and POD basis is ceaselessly renewed, the rules of development and change of future physical system would be very simulated well.

4.2 Implementation of Algorithm for Problem V

In the following, we give the implementation of algorithm for solving Problem V, which consists of seven steps.

Step 1 Generate the snapshots ensemble

$$W_i(x,z) = u_h^i,\ i = 1, 2, \cdots, L \ll N,$$

which may be the first L solutions for Problem IV, physical system trajectories by drawing samples from experiments and interpolation (or data assimilation), or previous results;

Step 2 Generate the correlation matrix $\boldsymbol{A} = (A_{ik})_{L\times L}$, $A_{ik} = \frac{1}{L}(W_i, W_k)_H$, and $(W_i, W_k)_H = (\nabla u_h^i, \nabla u_h^k)$, $(\cdot,\cdot)$ is the L^2-inner product;

Step 3 Solving the eigenvalue problem

$$\boldsymbol{A}\boldsymbol{v} = \lambda\boldsymbol{v},\quad \boldsymbol{v} = (a_1, a_2, \cdots, a_L)^T,$$

obtains eigenvectors $\boldsymbol{v}^k = (a_1^k, a_2^k, \cdots, a_L^k)$ and corresponding eigenvalues λ_k $(k = 1, 2, \cdots, l = \dim\{W_1, W_2, \cdots, W_L\})$;

Step 4 For given error δ needed, decide on the amounts m of degree of polynomial and d of POD basis such that $\tau^2 + h^{m+1} + \Big(\tau \sum\limits_{j=d+1}^{l} \lambda_j\Big)^{1/2} \leqslant \delta$;

Step 5 Generate POD basis $\psi_k(x,y)$:

$$\psi_k(x,y) = \frac{1}{\sqrt{L\lambda_k}} \sum_{i=1}^{L} a_i^k W_i(x,y) = \frac{1}{\sqrt{L\lambda_k}} \sum_{i=1}^{L} a_i^k u_h^i,\ k = 1, 2, \cdots, d.$$

Step 6 Taking $H^d = \mathrm{span}\,\{\psi_1(x,y), \psi_2(x,y), \cdots, \psi_d(x,y)\}$ and solving Problem V which only includes d degrees of freedom yield the solutions u_d^n $(n = 1, 2, \cdots, L, L+1, \cdots, N)$.

Step 7 If $\|u_d^{n-1} - u_d^n\|_0 \geqslant \|u_d^n - u_d^{n+1}\|_0$ $(n = L, L+1, \cdots, N-1)$, u_d^n $(n = 1, 2, \cdots, N)$ are the solutions for Problem V whose errors are greater than $\tau^2 + h^{m+1} + \Big(\tau \sum\limits_{j=d+1}^{l} \lambda_j\Big)^{1/2}$. Else, i.e., if $\|u_d^{n-1} - u_d^n\|_0 < \|u_d^n - u_d^{n+1}\|_0$ $(n = L, L+1, \cdots, N-1)$, let $W_i = u_d^i$ $(i = n-L, n-L-1, \cdots, n)$, repeat Step 1 to Step 6.

5 Some Numerical Experiments

In this section, some numerical examples of second-order hyperbolic equations are used to validate the feasibility and efficiency of the reduced FE formulation, i.e., Problem V based on POD method.

For the sake of convenience, without loss of generality, herein we consider a second-order hyperbolic equations by choosing $\varepsilon = 1/\pi^2$, $\overline{\Omega} = [0,2]\times[0,2]$, total time $T = 0.4$, $\varphi_0(x,y) = \sin\pi x \sin\pi y$, $\varphi_1(x,y) = -2\sin\pi x\sin\pi y$, and $f(x,y,t) = 5\mathrm{e}^{-2t}\sin\pi x\sin\pi y$ in Problem I as an example, whose ideas and approaches could directly apply to numerical computations for second-order hyperbolic equations with real practical applied background.

We first divide the field $\overline{\Omega}$ into 2000×2000 small squares with side length $\triangle x = \triangle y = 0.001$, and then link the diagonal of the square to divide each square into two triangles in the same direction which consists of triangulation $\Im_h$. Thus $h = \sqrt{2}\times 0.001$. In order to make $\tau = O(h)$ to be satisfied, we take time step size as $\tau = 0.002$.

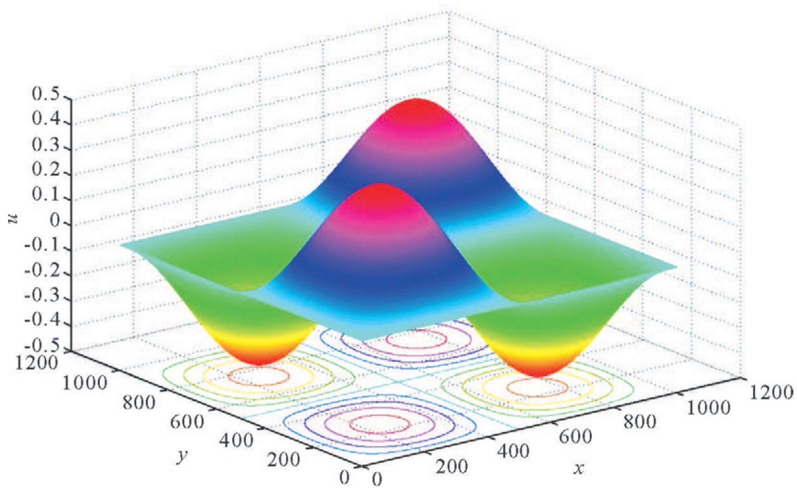

Fig.1 Figure of solution u_h^n of classical FE formulation when $t = 0.4$

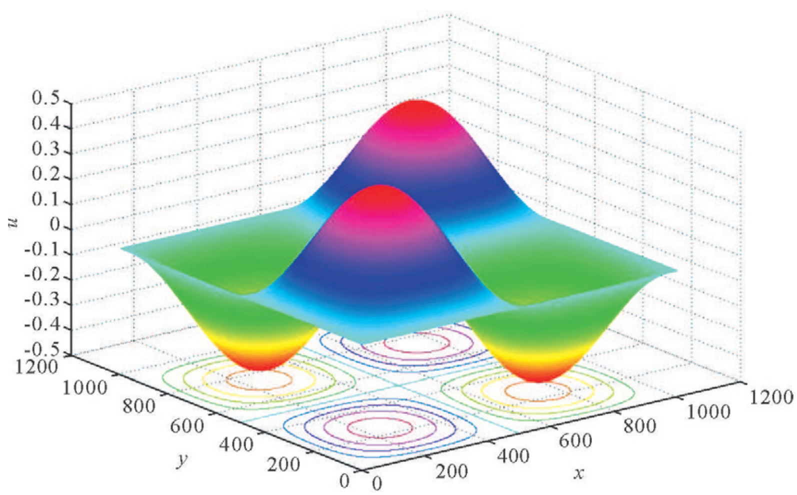

Fig.2 Figure of solution u_d^n of reduced FE formulation when $d = 6$, $t = 0.4$

We first find a set of numerical solutions u_h^n of classical FE formulation, i.e., Problem IV with piecewise polynomial of degree 1 when $n = 1, 2, \cdots, 200$, i.e., at time $t = 1\tau, 2\tau, \cdots, 200\tau$, constructing 200 numerical solutions u_h^n $(n = 1, 2, \cdots, 200)$. And then, the first 20 numerical solutions u_h^n $(n = 1, 2, \cdots, 20)$ are chosen as snapshots $W_i = u_h^i$ $(i = 1, 2, \cdots, 20)$. Finally, we find 20 eigenvalues which are arranged in a non-increasing order and 20 eigenvectors corresponding to then, and using (3.6), we construct 20 POD bases ψ_j $(j = 1, 2, \cdots, 20)$. Take the first 6 POD bases ψ_j $(j = 1, 2, \cdots, 6)$ from 20 POD bases ψ_j $(j = 1, 2, \cdots, 20)$ to expand into subspace H^d and compute a numerical solution at $t = 200\tau$ with reduced FE formulation, i.e., the reduced Problem V according seven steps in Subsection 4.2, without renewing POD basis, we obtain the numerical solutions u_d^n $(n = 200$, i.e., $t = 200\tau = 0.4)$ which are depicted graphically in Fig. 2. While the classical FE solution to Problem V with piecewise polynomial of degree 1 at $t = 200\tau = 0.4$, is depicted graphically in Fig. 1. Fig. 1 and Fig. 2 exhibit quasi-identical similarity, but POD solution is better than classical FE solution (due to using more initial numerical solutions, i.e., six POD bases, which is an open problem).

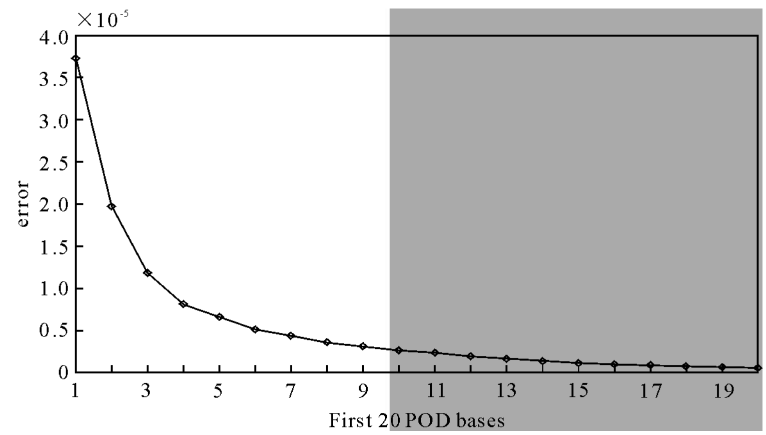

Fig.3 When $t = 0.4$, the errors between solutions of Problem V with different number of POD bases for a group of 20 snapshots and the classical FE solution of Problem IV with piecewise polynomial of degree 1

When we take 6 POD bases and $\tau = 0.002$, by computing we obtain that $\left[\tau \sum_{j=7}^{20} \lambda_j\right]^{1/2}$ $+\tau^2 + h^2 \leqslant 5 \times 10^{-6}$. Fig. 3 computationally shows the errors between solutions u_d^n of the reduced FE formulation with first 20 different POD bases among all POD bases and the solution u_h^n of classical FE formulation at $t = 200\tau$ (i.e., $n = 200$), respectively. Comparing the classical FE formulation with the reduced FE formulation containing 6 POD bases implementing the numerical simulation computations when total time $t = 200\tau$, we find that for classical FE formulation with piecewise linear polynomials for u_h^n, which has $2000 \times 2000 = 4 \times 10^6$ degrees of freedom, the required computing time is 240 seconds, while for the reduced FE formulation with 6 POD bases, which has only 6 degrees of freedom, the corresponding time is only 2 seconds, i.e., the required computing time to solve the classical FE formulation is as 120 times as that to do the reduced FE formulation with 6 POD bases, while the errors between their respective solutions do not exceed 5×10^{-6}. Though our examples are in sense recomputing what we have already computed by classical FE formulation (but only use the first 20 numerical solutions u_h^n $(n = 1, 2, \cdots, 20)$ in the first 20 steps), when we compute actual problems, we may also construct the snapshots and POD basis with interpolation or data assimilation by drawing samples from experiments, then solve directly the reduced FE formulation, while it is unnecessary to solve classical FE formulation such that the computational load could be alleviated and time-consuming of calculations in the computational process is saved. It also shows that finding the approximate solutions for second-order hyperbolic equations with the reduced FE formulation is computationally very effective. And the results for numerical examples are consistent with those obtained for the theoretical cases.

6 Conclusions

In this paper, we have employed the POD method to derive a reduced FE formulation for second-order hyperbolic equations with real practical applied background, analyzed the errors between the solutions of their classical FE formulation and the solutions of the reduced FE formulation based on POD method, and provide the steps of the implementation of algorithm for solving the reduced FE formulation, i.e., Problem V, which shows that our present method has improved and innovated the existing methods (for example, the methods in [10–20]). Comparing with the theoretical error estimates, the error estimates have been verified to provide quite good results, namely, the theoretical errors and the computing errors coincide within plot accuracy, thus validating both the feasibility and efficiency of our reduced FE formulation. Though snapshots and POD basis of our numerical examples are constructed with the first few solutions of the classical FE formulation, when one computes actual second-order hyperbolic equations, this process can be omitted in actual applications and one may construct the snapshots and POD basis with interpolation or data assimilation by drawing samples from experiments, then solving Problem V, while it is unnecessary to solve Problem IV such that the computational load could be alleviated and time-consuming of calculations in the computational process is saved. Therefore, the method in this paper gives a good prospect of extensive applications. Future research work in this area will aim at extending the reduced FE formulation, applying it to a set of more complicated PDEs such as the atmosphere quality forecast system and the ocean fluid forecast system.

References

[1] Dupont T. L^2 error estimates for Galerkin methods for second order hyperbolic equations. SIAM J Numer Anal, 1973, **10**: 880–889

[2] Yuan Y R, Wang H. Error estimates for the finite element methods of nonlinear hyperbolic equations. J Systems Sci Math Sci, 1985, **5**(3): 161–171

[3] Pani A K, Sinha R K. The efect of spatial quadrature on finite element Galerkin approximation to hyperbolic integro-differential equations. Numer Funct Anal Optim, 1998, **19**: 1129–1153

[4] Sinha R K. Finite element approximations with quadrature for second-order hyperbolic equations. Numer Methods PDE, 2002, **18**: 537–559

[5] Zhang J, Yang D. A splitting positive definite mixed element method for second-order hyperbolic equations. Numer Methods PDE, 2009, **25**(3): 622–636

[6] Holmes P, Lumley J L, Berkooz G. Turbulence, Coherent Structures, Dynamical Systems and Symmetry. Cambridge UK: Cambridge University Press, 1996

[7] Fukunaga K. Introduction to Statistical Recognition. New York: Academic Press, 1990

[8] Jolliffe I T. Principal Component Analysis. Berlin: Springer–Verlag, 2002

[9] Sirovich L. Turbulence and the dynamics of coherent structures: Part I-III. Quart Appl Math, 1987, **45**: 561–590

[10] Kunisch K, Volkwein S. Galerkin proper orthogonal decomposition methods for parabolic problems. Numer Math, 2001, **90**: 117–148

[11] Kunisch K, Volkwein S. Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics. SIAM J Numer Anal, 2002, **40**: 492–515

[12] Luo Z D, Xie Z H, Chen J. A reduced MFE formulation based on proper orthogonal decomposition for the non-stationary conduction-convection problems. Acta Math Sci, 2011, **31B**(5): 1765–1785

[13] Luo Z D, Zhu J, Wang R W, Navon I M. Proper orthogonal decomposition approach and error estimation of mixed finite element methods for the tropical Pacific Ocean reduced gravity model. Comp Methods Appl Mech Eng, 2007, **196**(41-44): 4184–4195

[14] Luo Z D, Chen J, Zhu J, Wang R W, Navon I M. An optimizing reduced order FDS for the tropical Pacific Ocean reduced gravity model. Int J Numer Methods Fluids, 2007, **55**(2): 143–161

[15] Luo Z D, Wang R W, Zhu J. Finite difference scheme based on proper orthogonal decomposition for the non-stationary Navier-Stokes equations. Science in China Series A: Mathematics, 2007, **50**(8): 1186–1196

[16] Luo Z D, Chen J, Nanvon I M, Yang X Z. Mixed finite element formulation and error estimates based on proper orthogonal decomposition for non-stationary Navier-Stokes equations. SIAM J Numer Anal, 2008, **47**(1): 1–19

[17] Luo Z D, Yang X Z, Zhou Y J. A reduced finite difference scheme based on singular value decomposition and proper orthogonal decomposition for Burgers equation. J Comput Appl Math, 2009, **229**(1): 97–107

[18] Luo Z D, Zhou Y J, Yang X Z. A reduced finite element formulation based on proper orthogonal decomposition for Burgers equation. Appl Numer Math, 2009, **59**(8): 1933–1946

[19] Sun P, Luo Z D, Zhou Y J. Some reduced finite difference schemes based on a proper orthogonal decomposition technique for parabolic equations. Appl Numer Math, 2010, **60**: 154–164

[20] Luo Z D, Chen J, Sun P, Yang X Z. Finite element formulation based on proper orthogonal decomposition for parabolic equations. Science in China Series A: Mathematics, 2009, **52**(3): 587–596

[21] Adams R A. Sobolev Space. New York: Academic Press, 1975

[22] Luo Z D. Mixed Finite Element Methods and Applications. Beijing: Science Press, 2006

[23] Ciarlet P G. The Finite Element Method for Elliptic Problems. Amsterdam: North-Holland, 1978

[24] Brezzi F, Fortin M. Mixed and Hybrid Finite Element Methods. New York: Springer-Verlag, 1991

[25] Girault V, Raviart P A. Finite Element Approximations of the Navier-Stokes Equations, Theorem and Algorithms. New York: Springer-Verlag, 1986

[26] Rudin W. Functional and Analysis. 2nd ed. New York: McGraw-Hill Co Inc, 1973