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Error Estimate of Nonconforming Finite Element Approximation for a Fourth Order Elliptic Variational Inequality

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Abstract In this paper, we discuss nonconforming finite element approximation for a fourth order elliptic variational inequality with a simply supported boundary and a constraint on mean curvature. Under appropriate assumptions on regularity, we obtain error estimate for the corresponding finite element method as follows: $\|u-u_h\|_h \leqslant ch^{1/2}(\|u\|_{3,\Omega}+\|f\|_{0,\Omega})$ for a kind of nonconforming finite elements including Morley's element, the first kind of Fraeijs de Veubeke's element and the incomplete biquadratic nonconforming plate element (WS's element).

Key Words and Phrases Variational Inequality; Nonconforming Element; Error Estimate

§ 1. Continuous Problem

Let Ω be a bounded domain of R^2 with a reasonably smooth boundary $\partial\Omega$. We consider the fourth order elliptic variational inequality related to a simply supported plate with a constraint on mean curvature:

$$\text{Fine } u \in K, \text{ such that } a(u, v-u) \geqslant (f, v-u), \qquad \forall\, v \in K, \tag{1}$$

where the bilinear form $a(u,v) = (\Delta u, \Delta v)$, $(v,w) = \int_\Omega vw\,dx$, $f \in L^2(\Omega)$, and the set K is defined by

$$K = \{v \in V = H_0^1(\Omega) \cap H^2(\Omega);\ \alpha \leqslant \Delta u \leqslant \beta \text{ a.e. on } \Omega\},$$

with $\alpha, \beta \in R$, $\alpha < 0 < \beta$.

For the problem (1), the bilinear form $a(\cdot,\cdot)$ is continuous and coercive on V, f is a linear and continuous functional on V and the set K is a closed convex and nonempty subset of V. Hence the problem (1) has a unique solution $u \in K$.

Lemma The solution u of the variational inequality (1) satisfies

$$\Delta u = \tau(F), \tag{2}$$

where $\tau(t)$ is the truncation:

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$$\tau(t)=\begin{cases}\alpha, & \text{if } t<\alpha;\\ t, & \text{if } \alpha\leqslant t\leqslant\beta;\\ \beta, & \text{if } t>\beta,\end{cases}$$

and F is a function satisfying

$$F\in H_0^1(\Omega),\quad \Delta F=f \qquad \text{in } \Omega. \tag{3}$$

If $f\in L^2(\Omega)$ then $u\in H_0^1(\Omega)\cap H^3(\Omega)$ and $\Delta u\in H_0^1(\Omega)$.

Properties of solutions of a fourth order elliptic variational inequality with a variety of constraints have been studied in [1]—[4], and their numerical analyses have been developed in [5]—[12]. In particular, the mixed finite element approximation for the problem (1) was discussed in [10]. In this paper, we discuss nonconforming finite element approximation with a kind of nonconforming plate element which includes Morley's element, the first kind of Fraeijs de Veubeke's element, and the incomplete biquadratic nonconforming plate element (WS's element), and obtain the following error estimate under appropriate assumptions on regularity:

$$\|u-u_h\|_h\leqslant ch(\|u\|_{3,\Omega}+\|f\|_{0,\Omega}).$$

§ 2. Error Estimate of Nonconforming Finite Element Approximation

To discuss nonconforming finite element approximation for the problem (1), we introduce a bilinear form $\tilde a(\cdot,\cdot)$. For arbitrary $v,w\in H_0^1(\Omega)\cap H^2(\Omega)$, let

$$\tilde a(v,w)=\int_\Omega\{\Delta v\Delta w-(\partial_{11}v\partial_{22}w+\partial_{22}v\partial_{11}w-2\partial_{12}v\partial_{12}w)\}\mathrm{d}x,$$

where $\partial_{ij}v$ and $\partial_{ij}w$ denote $\frac{\partial^2 v}{\partial x_i\partial x_j}$ and $\frac{\partial^2 w}{\partial x_i\partial x_j}$ $(1\leqslant i,j\leqslant 2)$ respectively.

For arbitrary $v,w\in H_0^1(\Omega)\cap H^2(\Omega)$, we have

$$\int_\Omega(\partial_{11}v\partial_{22}w+\partial_{22}v\partial_{11}w-2\partial_{12}v\partial_{12}w)\mathrm{d}x=-\int_{\partial\Omega}(\partial_{ss}v\partial_\nu w-\partial_{s\nu}v\partial_s w)\mathrm{d}s=0,$$

where $\partial_s v$ and $\partial_\nu v$ denote the tangential and normal derivatives respectively.

Thus, for arbitrary $v,w\in H_0^1(\Omega)\cap H^2(\Omega)$, we have

$$\tilde a(v,w)=a(v,w).$$

Then we can rewrite the continuous problem (1) as follows:

$$\text{Find } u\in K, \text{ such that } \tilde a(u,v-u)\geqslant(f,v-u),\qquad \forall\, v\in K. \tag{4}$$

In order to give the finite element approximation for the problem (1), we suppose for simplicity that Ω is a bounded polygonal domain of R^2. Let $\{\mathcal{T}_h\}_h$ be a regular family of triangulation of Ω made up of triangles or rectangulars $T\in\mathcal{T}_h$ and $\bar\Omega=\bigcup_{T\in\mathcal{T}_h}\bar T$. We approximate $V=H_0^1(\Omega)\cap H^2(\Omega)$ by using a nonconforming space V_h. For arbitrary $v_h\in V_h$, we assume that

1° The degrees of freedom of v_h at vertices of $T\in\mathcal{T}_h$ include displacement;

2° The displacement of the nodes at $\partial\Omega$ takes zero so that v_h satisfies the simply supported condition.

We approximate the convex set K and the bilinear from $a(\cdot,\cdot)$ by

$$K_h = \{v_h \in V_h;\ \alpha \leqslant \frac{1}{\text{meas}T}\int_T \Delta v_h \mathrm{d}x \leqslant \beta,\ \forall\ T \in \mathscr{T}_h\},$$

and

$$\tilde{a}_h(v_h, w_h) = \sum_{T\in\mathscr{T}_h}\int_T \sum_{|\alpha|=2} D^\alpha v_h D^\alpha w_h \mathrm{d}x, \qquad \forall\ v_h, w_h \in V_h,$$

respectively.

We mark

$$|v_h|_{k,h} = \Big(\sum_{T\in\mathscr{T}_h} |v_h|^2_{k,T}\Big)^{1/2} \quad (k = 0,1,2),$$

$$\|v_h\|_h = |v_h|_{2,h}, \qquad \forall\ v_h \in V_h \cup H^2(\Omega),$$

where

$$|v_h|_{k,T} = \Big(\int_T \sum_{|\alpha|=k} |D^\alpha v_h|^2 \mathrm{d}x\Big)^{1/2}$$

and suppose that $\|v_h\|_h$ is a norm over V_h.

The corresponding finite element approximation problem for the variational problem (1) is

Find $u_h \in K_h$, such that $a_h(u_h, v_h - u_h) \geqslant (f, v_h - u_h)$, $\qquad \forall\ v_h \in K_h$. (5)

which has one and only one solution $u_h \in K_h$.

In order to estimate the error of the approximation solution we denote the set of edges of $T \in \mathscr{T}_h$ which is not on the boundary $\partial\Omega$ by S_h^*, the set of edges located on $\partial\Omega$ by G_h^*, and the jump of w along $s \in S_h^*$ by $[w]$, where the jump $[w]$ of a edge s of $T \in \mathscr{T}_h$ ($s \in S_h^*$) is from its inner part toward its outer part. We set $[w] = w$ along $s \in G_h^*$. Moreover, we assume for V_h that

(C_1) There exists a linear operator r_h: $H^3(\Omega) \cap K \to K_h$, such that

$$|v - r_h v|_{k,h} \leqslant ch^{3-k}\|v\|_{3,\Omega} \qquad (k = 0,1,2);$$

(C_2) $\Big|\sum_{T\in\mathscr{T}_h}\int_{\partial T} w\partial_\nu v_h \mathrm{d}s\Big| \leqslant ch\|w\|_{1,\Omega}\|v_h\|_h$, $\forall\ w \in H_0^1(\Omega)$, $v_h \in V_h$;

where C is a constant which is independent of h and may takes different values in different places.

Under the above assumptions on V_h and K_h, we have the following error estimate theorem for the nonconforming finite element approximation problem (5).

Theorem Suppose that u and u_h are solutions of the continuous problem (1) and the discrete problem (5) respectively, $f \in L^2(\Omega)$, $\Delta u \in H_0^1(\Omega)$, and $u \in H^3(\Omega)$. Under the above assumptions on V_h and K_h and the conditions (C_1) and (C_2), we have the following error estimate:

$$\|u - u_h\|_h \leqslant ch(\|u\|_{3,\Omega} + \|f\|_{0,\Omega}). \tag{6}$$

Proof For any $v_h \in K_h$, we know by (5) that

$$\|v_h - u_h\|_h^2 = a_h(v_h - u_h,\ v - u_h)$$

$$= a_h(v_h - u,\ v_h - u_h) + a_h(u - u_h, v_h - u_h)$$

$$\leqslant C(\|u - r_h\|_h \|v_h - u_h\|_h + a_h(u, r_h - u_h) - (f, r_h - u_h)). \tag{7}$$

Now we analyse the last two terms of the right hand side of (7). We have

$$\begin{aligned}
&a_h(u, r_h u - u_h) - (f, r_h u - u_h) \\
&= (\Delta u - F, \Delta(r_h u - u_h))_h + \sum_{T \in \mathcal{T}_h} \int_{\partial T} (u_{nn}\partial_s(r_h u - u_h) - u_{ns}\partial_s(r_h u - u_h))\mathrm{d}s \\
&\quad + \Big\{ \sum_{T \in \mathcal{T}_h} \int_{\partial T} F\partial_n(r_h u - u_h)\mathrm{d}s - \sum_{T \in \mathcal{T}_h} \int_{\partial T} \partial_n F(r_h u - u_h)\mathrm{d}s \Big\} \\
&\equiv I_1 + I_2 + I_3,
\end{aligned} \tag{8}$$

where $(\cdot, \cdot)_h = \sum_{T \in \mathcal{T}_h} (\cdot, \cdot)_T$.

Setting

$$w = F - r(F), \quad w^+ = \sup\{0, w\}, \quad w^- = \sup\{0, -w\},$$

we have

$$w = w^+ - w^-, \quad w^+, w^- \in H_0^1(\Omega). \tag{9}$$

By the lemma and (9) we obtain

$$w^+(\Delta u - \beta) = 0, \quad w^-(\alpha - \Delta u) = 0. \tag{10}$$

$$\begin{aligned}
I_1 &= (\Delta u - F, \quad \Delta(r_h u - u_h))_h \\
&= (\Delta u - F, \Delta(r_h u - u_h))_h (\Delta u - F, \Delta(u - u_h))_h.
\end{aligned} \tag{11}$$

Using Green's formula we have

$$\begin{aligned}
&(\Delta u - F, \Delta(r_h u - u))_h \\
&= -(\nabla(\Delta u - F), \nabla(r_h u - u))_h + \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\Delta u - F)\partial_n(r_h u - u)\mathrm{d}s.
\end{aligned} \tag{12}$$

From $\Delta u - F \in H_0^1(\Omega)$ and (C_1) we deduce that

$$\begin{aligned}
&\Big| \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\Delta u - F)\partial_n(r_h u - u)\mathrm{d}s \Big| \\
&= \Big| \sum_{\tau \in \mathcal{E}_h^e} \int_{\tau} (\Delta u - F)[\partial_n(r_h u - u)]\mathrm{d}s \Big| \\
&\leqslant ch(\|u\|_{3,\Omega} + \|f\|_{0,\Omega})\|r_h u - u\|_h \\
&\leqslant ch^2(\|u\|_{3,\Omega} + \|f\|_{0,\Omega})\|u\|_{3,\Omega}.
\end{aligned} \tag{13}$$

We obtain from (12) and (13) that

$$(\Delta u - F, \Delta(r_h u - u))_h \leqslant ch^2(\|u\|_{3,\Omega} + \|f\|_{0,\Omega})\|u\|_{3,\Omega}. \tag{14}$$

It follows from (10) and $u_h \in K_h$ that

$$(w^+, -\Delta(u - u_h))_h = (w^+, -\Delta u + \beta)_h + (w^+, \Delta u - \beta)_h \leqslant 0, \tag{15}$$

and

$$(w^-, -\Delta(u - u_h))_h = (w^-, \Delta u - \alpha)_h + (w^-, \alpha - \Delta u_h)_h \leqslant 0. \tag{16}$$

Hence we have

$$\begin{aligned}
&(\Delta u - F, \Delta(r_h u - u_h))_h \\
&= (w^+, -\Delta(u - u_h))_h + (w^-, -\Delta(u - u_h))_h \leqslant 0.
\end{aligned} \tag{17}$$

It follows from (11), (14) and (17) that

$$I_1 \leqslant ch^2(\|u\|_{3,\Omega} + \|f\|_{0,\Omega})\|u\|_{3,\Omega}. \tag{18}$$

We clearly see that

$$\begin{aligned} I_2 &= \sum_{T\in\mathscr{T}_h} \int_{\partial T} (u_{ns}\partial_s(r_h u - u_h) - u_{ss}\partial_n(r_h u - u_h))\mathrm{d}s \\ &= \sum_{s\in S_h^*\cup G_h^*} \int_s u_{ns}[\partial_s(r_h u - u_h)]\mathrm{d}s - \sum_{T\in\mathscr{T}_h} \int_{\partial T} u_{ss}\partial_n(r_h u - u_h))\mathrm{d}s \\ &\equiv J_1 + J_2. \end{aligned} \tag{19}$$

Let $\pi \circ U_{ns}$ denote the integral mean value of u_{ns} along $s\in S_h^* \bigcup G_h^*$. By using that $w_h=0$ at the endpoints of $s\in G_h^*$ and the continuity of w_h at the endpoints of $s\in S_h^*$, we clearly obtain (where $w_h = r_h u - u_h$)

$$\begin{aligned} J_1 &= \sum_{s\in S_h^*\cup G_h^*} \int_s (u_{ns} - \pi \circ u_{ns})[\partial_s(r_h u - u_h)]\mathrm{d}s \\ &\leqslant ch\|u\|_{3,\Omega}\|r_h u - u_h\|_h. \end{aligned} \tag{20}$$

It follows from (C_2) that

$$J_2 \leqslant ch\|u\|_{3,\Omega}\|r_h u - u_h\|_h. \tag{21}$$

Hence we have from (19)—(21) that

$$I_2 \leqslant ch\|u\|_{3,\Omega}\|r_h u - u_h\|_h. \tag{22}$$

It follows from (3) and (C_2) that

$$\left|\sum_{T\in\mathscr{T}_h} \int_{\partial T} F\partial_n(r_h u - u_h)\mathrm{d}s\right| \leqslant ch\|f\|_{0,\Omega}\|r_h u - u_h\|_h. \tag{23}$$

Denoting the linear or bilinear interpolation of w_h for the division $\mathscr{T}_h$ by $P_h w_h$, we then have

$$\begin{aligned} &-\sum_{T\in\mathscr{T}_h} \int_{\partial T} \partial_n F(r_h u - u_h)\mathrm{d}s \\ &= -\sum_{T\in\mathscr{T}_h} \int_{\partial T} \partial_n F\{(r_h u - u_h) - P_h(r_h u - u_h)\}\mathrm{d}s \\ &\leqslant C(h|F|_{1,\Omega} + h^2|F|_{2,\Omega})\|r_h u - u_h\|_h \\ &\leqslant ch\|f\|_{0,\Omega}\|r_h u - u_h\|_h. \end{aligned} \tag{24}$$

It follows from (23) and (24) that

$$I_3 \leqslant ch\|f\|_{0,\Omega}\|r_h u - u_h\|_h, \tag{25}$$

which togather with (7), (8), (18), (19) and (25) implies that (6) holds. The proof is complete.

Remark If there exists an interpolation operator r_h: $H_0^1(\Omega)\bigcap H^3(\Omega)\to V_h$ such that

$$\int_{\partial T} \partial_n v\mathrm{d}s = \int_{\partial T} \partial_n(r_h v)\mathrm{d}s, \qquad \forall v \in H_0^1(\Omega) \cap H^3(\Omega),\ \forall T \in \mathscr{T}_h, \tag{26}$$

and

$$|v - r_h v|_{k,h} \leqslant ch^{3-k}\|v\|_{3,\Omega} \qquad (k = 0,1,2),\ \forall v \in H_0^1(\Omega) \cap H^3(\Omega), \tag{27}$$

then by Green's formula we know that if $v\in K$ then $r_h v\in K_h$. Therefore (C_1) holds. Moreover, if we assume that

$$\int_s [\partial_\nu v_h] ds = 0, \qquad \forall\, s \in S_h^* . \tag{28}$$

then (C_2) also holds. Hence we have:

$$\|u - u_h\|_h \leqslant ch(\|u\|_{3,\Omega} + \|f\|_{0,\Omega}).$$

§ 3. Examples

Example 1 Let $\mathscr{T}_h$ be a regular triangulation and $\bar{\Omega} = \bigcup_{T \in \mathscr{T}_h} \bar{T}$. We approximate $V = H_0^1(\Omega) \cap H^2(\Omega)$ by the nonconforming Morley's element space V_h which is defined by

$$V_h = \{v_h \in L^2(\Omega);\ v_h|_T \in P(2),\ \forall\, T \in \mathscr{T}_h\},$$

where $v_h \in V_h$ is continuous at the vertices of $\mathscr{T}_h$ and vanishes at vertices which belong to $\partial\Omega$, and its first outer normal derivatives are continuous at the midpoints of the edges of a triangle $T \in \mathscr{T}_h$ (refer to Figure 1). For more details, see [8,13]. Define an operator r_h as follows:

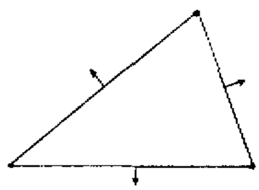

Figure 1

$$\begin{cases} v \in H^3(\Omega) \cap K \to r_h v \in K_h, \\ r_h v(p) = v(p) & \text{at } p \text{ vertex of } \mathscr{T}_h, \\ \int_s \partial_\nu r_h v ds = \int_s \partial_\nu v ds, & \forall\, s \text{ edge of } \mathscr{T}_h. \end{cases} \tag{29}$$

By [13] we know that for any $v_h \in V_h$, the condition of the Remark is satisfied. And therefore, for Morley's element we have

$$\|u - u_h\|_h \leqslant ch(\|u\|_{3,\Omega} + \|f\|_{0,\Omega}).$$

Example 2 Let $\mathscr{T}_h$ be a regular triangulation, and $\bar{\Omega} = \bigcup_{T \in \mathscr{T}} \bar{T}$. We approximate $V = H_0^1(\Omega) \cap H^2(\Omega)$ by the first kind of Fraeijs de Veubeke's element space V_h defined by [13] whose degrees of freedom are the values of the function at the midpoints of the edges of the element and the mean value of the first outer normal derivative along each edge. $v_h = 0$ at endpoints and midpoint of $s \in G_h^*$ for $v_h \in V_h$ (refer to Figure 2). For more details, see [13]. Define an operator r_h as follows:

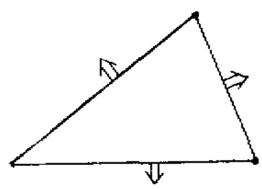

Figure 2

$$\begin{cases} v \in H^3(\Omega) \cap K \to r_h v \in K_h, \\ r_h v(p) = v(p) \text{ at } p \text{ vertex or midpoint of edges of } T \in \mathscr{T}_h, \\ \int_s \partial_\nu (r_h v) ds = \int_s \partial_\nu v ds \quad \forall\, s \in S_h^* \cup G_h^*. \end{cases} \tag{30}$$

By [13] we see that the condition of the remark is satisfied. And hence we obtain the following estimate for the corresponding finite element discrete problem:

$$\|u-u_h\|_h \leqslant ch(\|u\|_{3,\Omega}+\|f\|_{0,\Omega}).$$

Example 3 Let $\mathscr{T}_h$ be a regular rectangular division, and $\overline{\Omega}=\bigcup_{T\in\mathscr{T}_h}\overline{T}$. We approximate $V=H_0^1(\Omega)\bigcap H^2(\Omega)$ by the incomplete biquadratic nonconforming plate element (WS's element) space V_h (which was studied by Wu Maoqing[14] and Shi Zhongci[15] respectively). The shaped function v_h on T is defined as follows:

$$V_h=a_1+a_2x+a_3y+a_4x^2+a_5xy+a_6y^2+a_7x^2y+a_8xy^2,$$

which has eight degrees of freedom a_i $(1\leqslant i\leqslant 8)$ determined by v_h at vertices of $T\in\mathscr{T}_h$ and $\partial_\nu v_h$ at midpoints of $s\in S_h^*\bigcup G_h^*$ (refer to Figure 3). For more details, see [11,14,15]. The degrees of freedom of displacement located on $\partial\Omega$ are zero. We define linear operator r_h as follows:

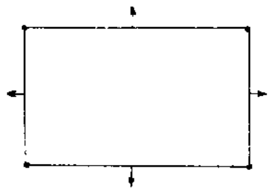

Figure 3

$$\begin{cases} v\in H^3(\Omega)\bigcap K\rightarrow r_hv\in K_h, \\ r_hv(p)=v(p) \text{ at vertices of } T\in\mathscr{T}_h, \\ \int_s\partial_\nu r_hv\,ds=\int_s\partial_\nu v\,ds, \quad \forall\, s\in S_h^*\bigcup G_h^*. \end{cases} \quad (31)$$

By [11,12,15] we can show that both (C_1) and (C_2) hold. Hence, for the incomplete biquadratic nonconforming plate element space we have

$$\|u-u_h\|_h \leqslant ch(\|u\|_{3,\Omega}+\|f\|_{0,\Omega}).$$

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